Ternary Interpolatory Subdivision

by

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Abstract

Ternary Interpolatory Subdivision

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Subdivision is an important and efficient tool for rendering smooth curves and surfaces in computer graphics, by repeatedly applying a subdivision (refining) scheme to a given set of points. In the literature, attention has been mostly restricted to developing binary subdivision schemes. The primary emphasis of this thesis is on ternary subdivision, and in particular on the interpolatory case. We will derive a symmetric ternary interpolatory subdivision scheme for the rendering of curves, satisfying analogous properties to the Dubuc-Deslauriers binary scheme. Explicit construction methods, as well as a corresponding convergence analysis, will be presented. Graphical illustrations of the results will also be provided.

Uittreksel

Ternêre Interpolerende Subdivisie

("Ternary Interpolatory Subdivision")

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Subdivisie bied 'n belangrike en doeltreffende metode om gladde krommes en oppervlakke in rekenaargrafika te genereer. Hierdie metode behels dat 'n subdivisieskema (of verfyningskema) herhaaldelik toegepas word op 'n gegewe versameling punte. In die literatuur word daar hoofsaaklik gefokus op die ontwikkeling van binêre subdivisieskemas. In hierdie tesis word die klem gelê op ternêre subdivisieskemas, en in die besonder op interpolerende skemas. Ons sal 'n simmetriese ternêre interpolerende subdivisieskema, wat analoë eienskappe as dié van die Dubuc-Deslauriers binêre skema bevredig, ontwikkel, om krommes te lewer. Eksplisiete konstruksiemetodes en ooreenkomstige konvergensieanalise, asook grafiese illustrasies van die resultate, sal getoon word.

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Soli Deo Gloria.

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Chapter 1

Introduction

1.1 Overview

Subdivision is an important and efficient tool for rendering smooth curves and surfaces, by repeatedly applying a subdivision (refining) scheme to a given set of points. It has a wide range of applications, for example in computer animation and graphics (see [9] and [8]), and in wavelets analysis (see [1]). Interpolatory subdivision schemes allow the user to retain the initial set of points at each iteration in the scheme. Applications range from the design of free-form surfaces and scattered data interpolation, to high quality rendering and mesh generation, for example in finite element analysis (see [15]). In this thesis, we will study interpolatory subdivision schemes for the rendering of curves.

Binary subdivision refers to subdivision schemes with refinement factor 2, that is, the number of points at each iteration is roughly doubled at each iteration. In the literature, attention has initially been mostly restricted to developing binary schemes. We will focus our attention on ternary schemes, with refinement factor 3, that is, the number of points at each iteration is roughly tripled at each iteration.

In 1987, S. Dubuc and G. Deslauriers introduced a symmetric binary interpolatory subdivision scheme (see [11] and [10]), which satisfies a polynomial filling property up to a given degree. Further analysis and formulations of this scheme are given in [6], [5], [4] and [1]. In this thesis, we derive a symmetric ternary interpolatory subdivision scheme, satisfying analogous properties to the Dubuc-Deslauriers binary scheme.

Convergence of a subdivision scheme is an important concept. Convergence of general (not necessarily interpolatory) binary subdivision schemes was analysed in [1], as well as in [12], which was adapted for ternary schemes in [14] and [13]. In this thesis, we derive convergence criteria for specifically ternary interpolatory subdivision schemes, by adapting and extending the results for binary interpolatory schemes in [17].

After establishing notation in the remainder of Chapter 1, Chapter 2 will be devoted to establishing results on 3-refinability and ternary subdivision, on which we will rely in the chapters to follow, with specific reference to analogous results on 2-refinability and binary subdivision in [1]. We will also give a precise definition of the notion of convergence in subdivision schemes, and derive properties of the limit function obtained.

In Chapter 3, we will derive a symmetric ternary interpolatory subdivision scheme, analogous to the Dubuc-Deslauriers binary scheme. We will do this by first assuming that such a subdivision scheme, with an interpolatory refinable basis function, exists, and then obtaining necessary conditions to be satisfied by the refinement sequence of the refinable function. We will then complete the argument in Chapter 4 by deriving a convergence criterion on the refinement sequence, which, when satisfied, will ensure the convergence of the corresponding subdivision scheme, and thereby proving the existence of an interpolatory refinable basis function for the subdivision limit curve.

In Chapter 5, we will start by presenting an algorithm for the rendering of closed curves by ternary interpolatory subdivision. Secondly, we will apply the results obtained in the previous chapters in specific examples with illustrations, and compare these results with analogous results in binary interpolatory subdivision.

Conclusions will follow in Chapter 6.

1.2 Notation

(i) We denote by $\mathbb{N}$ the set of natural numbers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers. We also denote by $\mathbb{R}^k$ the set

$$\{\boldsymbol{x} = (x_1, x_2, \ldots, x_k) : x_j \in \mathbb{R}, j = 1, \ldots, k\}.$$

(ii) We denote by $C(\mathbb{R})$ the space of all real-valued continuous functions on $\mathbb{R}$.

(iii) We write $C_0 = C_0(\mathbb{R})$ for the subspace of functions f in $C(\mathbb{R})$, such that f vanishes identically outside some bounded interval, in which case f will be called a compactly supported continuous function (on $\mathbb{R}$).

(iv) We denote by $C^n = C^n(\mathbb{R})$ the collection of functions which, together with their derivatives up to order n, are in $C(\mathbb{R})$, where $C^0 = C(\mathbb{R})$, and we define $C_0^n := C_0 \cap C^n$.

(v) We write π_k for the linear space of polynomials of degree $\leq k$.

(vi) We denote by $l(\mathbb{Z})$ the space of all bi-infinite sequences defined on the set $\mathbb{Z}$ of all integers, with $\boldsymbol{c} = \{c_j\} \in l(\mathbb{Z})$ if and only if $c_j \in \mathbb{R}^s$, $j \in \mathbb{Z}$, for $s = 1, 2, 3$.

(vii) We denote by $l_0 = l_0(\mathbb{Z})$ the subspace of sequences in $l(\mathbb{Z})$ with only finitely many non-zero elements.

(viii) We denote by $l^\infty = l^\infty(\mathbb{Z})$ the subspace of sequences in $l(\mathbb{Z})$ that are bounded, that is, $\{c_j\} \in l^\infty(\mathbb{Z})$ if and only if

$$\sup_j |c_j| < \infty.$$

(ix) Suppose $g \in C_0$. Let $\mu := \inf\{x : g(x) \neq 0\}$ and $\nu := \sup\{x : g(x) \neq 0\}$, so that $g(x) = 0$ for $x \leq \mu$ or $x \geq \nu$. Then we denote the closure of the convex hull of the support of g by

$$\mathrm{supp}^c g = [\mu, \nu]\,.$$

(x) Suppose $\{c_j\} \in l_0$. Let μ and ν be the largest and smallest integers, respectively, for which $c_j = 0$ for all $j < \mu$ or $j > \nu$. Then we denote the support of $\{c_j\}$ by

$$\mathrm{supp}\,\{c_j\} := [\mu, \nu] \cap \mathbb{Z} = [\mu, \nu]\big|_{\mathbb{Z}}.$$

(xi) We denote by $||\cdot||_\infty$ the infinity norm (or sup norm) as in

$$||\boldsymbol{c}||_\infty := \sup_{j \in \mathbb{Z}} |c_j|, \quad \boldsymbol{c} = \{c_j\} \in l^\infty,$$

or

$$||f||_\infty := \sup_{x \in \mathbb{R}} |f(x)|, \quad f \in C(\mathbb{R}).$$

(xii) We define

$$\sum_j := \sum_{j \in \mathbb{Z}}.$$

(xiii) For any non-negative integer k, we define

$$\binom{k}{j} := \begin{cases} \frac{k!}{j!(k-j)!}, & j = 0, \ldots, k, \\ 0, & j \notin \{0, \ldots, k\}\,, \end{cases}$$

where $0! := 1$.

(xiv) For $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the largest integer $\leq x$, while $\lceil x \rceil$ denotes the smallest integer $\geq x$.

Chapter 2

Subdivision and refinability

The integer shifts of refinable functions, that is, functions that can be generated from finitely many integer shifts of their scaled formulation, form basis functions for the limit curve obtained from the subdivision process. This chapter is devoted to the study of the concepts of refinability, subdivision and subdivision convergence. We will define these concepts and prove some general results, on which we will rely in later chapters.

2.1 Refinability and scaling functions

We start by giving a formal definition of a refinable function.

Definition 2.1.1 *Let ϕ be a function in C_0, and let $\{p_j\}$ be a sequence in l_0. For any integer $k \geq 2$, ϕ is called a k-refinable function if*

$$\phi(x) = \sum_j p_j \phi\left(kx - j\right), \quad x \in \mathbb{R}. \tag{2.1.1}$$

The sequence $\{p_j\}$ is called the corresponding refinement sequence, and (2.1.1) is called a refinement relation (or equation).

If, in addition to (2.1.1), ϕ satisfies the unit integral condition

$$\int_{-\infty}^{\infty} \phi(x)dx = 1, \tag{2.1.2}$$

then the k-refinable function ϕ is called a k-scaling function.

As mentioned in Chapter 1, we will focus our attention on 3-refinable functions, that is, functions $\phi \in C_0$ satisfying the refinement relation

$$\phi(x) = \sum_j p_j \phi\left(3x - j\right), \quad x \in \mathbb{R}, \tag{2.1.3}$$

for some sequence $\{p_j\} \in l_0$. Results obtained from here on will refer to 3-refinable functions and ternary subdivision schemes.

We also define the notion "partition of unity", as follows.

Definition 2.1.2 *Let $g \in C_0$. Then g is said to provide a partition of unity if*

$$\sum_j g(x-j) = 1, \quad x \in \mathbb{R}. \tag{2.1.4}$$

Note that $\sum_j$ is a finite sum, since $\text{supp}^c g$ is a compact interval.

We now give a few examples.

Example 2.1.3 Consider the box function

$$\chi_{[0,1)}(x) := \begin{cases} 1, & x \in [0,1); \\ 0, & x \notin [0,1), \end{cases} \tag{2.1.5}$$

that is, $\chi_{[0,1)}$ is the characteristic function on $[0,1)$. It is easy to check that

$$\chi_{[0,1)}(x) = (1)\chi_{[0,1)}(3x) + (1)\chi_{[0,1)}(3x-1) + (1)\chi_{[0,1)}(3x-2), \quad x \in \mathbb{R},$$

according to which $\chi_{[0,1)}$ satisfies the refinement relation (2.1.3), with

$$\{p_0, p_1, p_2\} = \{1,1,1\}; \quad p_j = 0, \ j \notin \{0,1,2\}.$$

However, since the condition $\chi_{[0,1)} \in C_0$, as required in Definition 2.1.1, is not satisfied, we do not consider $\chi_{[0,1)}$ to be a 3-refinable function.

The graph of $\chi_{[0,1)}(x)$ is given in Figure 2.1 (a), and the graphs of $\chi_{[0,1)}(3x)$, $\chi_{[0,1)}(3x-1)$ and $\chi_{[0,1)}(3x-2)$ are given in, respectively, Figures 2.1 (b), (c) and (d). ■

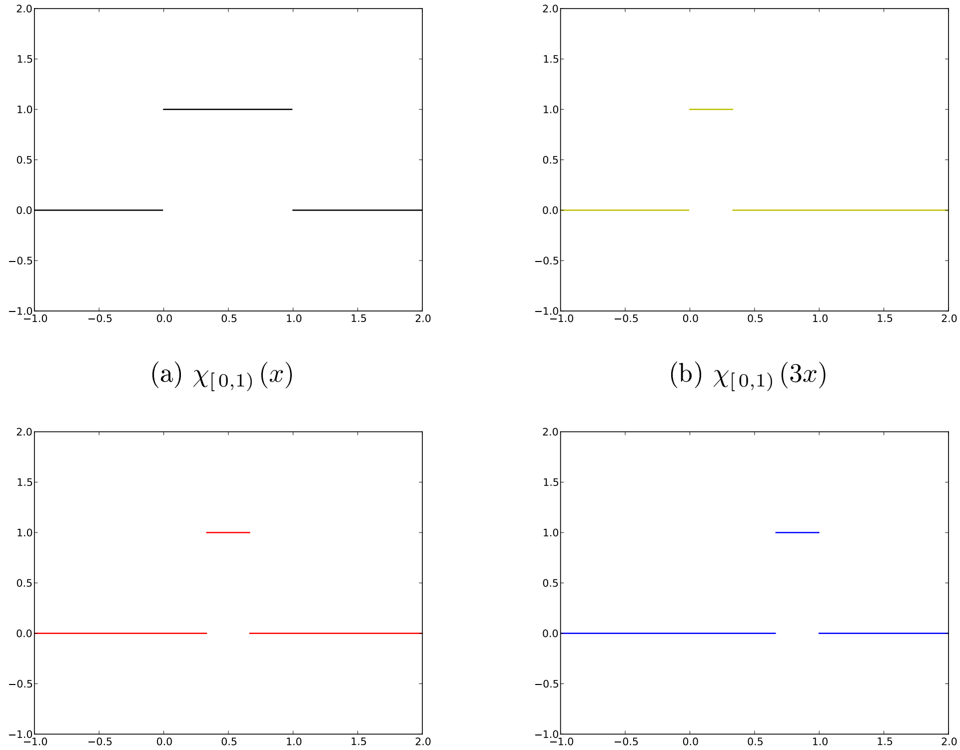
Example 2.1.4 Next, consider the hat function

$$h(x) := \begin{cases} 1+x, & -1 \le x < 0; \\ 1-x, & 0 \le x < 1; \\ 0, & x \in \mathbb{R} \notin [-1,1). \end{cases} \tag{2.1.6}$$

Note from (2.1.6) that

$$h(3x+2) = \begin{cases} 3+3x, & -1 \le x < -\frac{2}{3}; \\ -1-3x, & -\frac{2}{3} \le x < -\frac{1}{3}; \\ 0, & x \in \mathbb{R} \notin [-1,-\frac{1}{3}); \end{cases} \tag{2.1.7}$$

$$h(3x+1) = \begin{cases} 2+3x, & -\frac{2}{3} \le x < -\frac{1}{3}; \\ -3x, & -\frac{1}{3} \le x < 0; \\ 0, & x \in \mathbb{R} \notin [-\frac{2}{3},0); \end{cases} \tag{2.1.8}$$



(a) $\chi_{[0,1)}(x)$ (b) $\chi_{[0,1)}(3x)$

(c) $\chi_{[0,1)}(3x-1)$ (d) $\chi_{[0,1)}(3x-2)$

Figure 2.1: Refinability of the box function.

$$h(3x) = \begin{cases} 1+3x, & -\frac{1}{3} \leq x < 0; \\ 1-3x, & 0 \leq x < \frac{1}{3}; \\ 0, & x \in \mathbb{R} \notin [-\frac{1}{3}, \frac{1}{3}); \end{cases} \tag{2.1.9}$$

$$h(3x-1) = \begin{cases} 3x, & 0 \leq x < \frac{1}{3}; \\ 2-3x, & \frac{1}{3} \leq x < \frac{2}{3}; \\ 0, & x \in \mathbb{R} \notin [0, \frac{2}{3}); \end{cases} \tag{2.1.10}$$

$$h(3x-2) = \begin{cases} -1+3x, & \frac{1}{3} \leq x < \frac{2}{3}; \\ 3-3x, & \frac{2}{3} \leq x < 1; \\ 0, & x \in \mathbb{R} \notin [\frac{1}{3}, 1). \end{cases} \tag{2.1.11}$$

With the definition

$$F(x) := \tfrac{1}{3}h(3x+2) + \tfrac{2}{3}h(3x+1) + h(3x) + \tfrac{2}{3}h(3x-1) + \tfrac{1}{3}h(3x-2), \tag{2.1.12}$$

it then follows from (2.1.7) - (2.1.11) that:

for $-1 \leq x < -\frac{2}{3}$,

$$F(x) = \tfrac{1}{3}(3+3x) = 1+x; \tag{2.1.13}$$

for $-\frac{2}{3} \leq x < -\frac{1}{3}$,

$$F(x) = \tfrac{1}{3}(-1-3x) + \tfrac{2}{3}(2+3x) = 1+x; \tag{2.1.14}$$

for $-\frac{1}{3} \leq x < 0$,

$$F(x) = \tfrac{2}{3}(-3x) + (1+3x) = 1+x; \tag{2.1.15}$$

for $0 \leq x < \frac{1}{3}$,

$$F(x) = (1-3x) + \tfrac{2}{3}(3x) = 1-x; \tag{2.1.16}$$

for $\frac{1}{3} \leq x < \frac{2}{3}$,

$$F(x) = \tfrac{2}{3}(2-3x) + \tfrac{1}{3}(-1+3x) = 1-x; \tag{2.1.17}$$

for $\frac{2}{3} \leq x < 1$,

$$F(x) = \tfrac{1}{3}(3-3x) = 1-x, \tag{2.1.18}$$

whereas also

$$F(x) = 0, \quad x \in \mathbb{R} \setminus [-1,1). \tag{2.1.19}$$

According to (2.1.13) - (2.1.19), together with (2.1.12) and (2.1.6), the hat function h satisfies the identity

$$h(x) = \tfrac{1}{3}h(3x+2) + \tfrac{2}{3}h(3x+1) + h(3x) + \tfrac{2}{3}h(3x-1) + \tfrac{1}{3}h(3x-2), \quad x \in \mathbb{R}, \tag{2.1.20}$$

and thus, since also $h \in C_0$, a comparison of (2.1.3) and (2.1.20) shows that $\phi = h$ is a refinable function, with refinement sequence $\{p_j\}$ given by

$$\{p_{-2}, p_{-1}, p_0, p_1, p_2\} = \left\{\tfrac{1}{3}, \tfrac{2}{3}, 1, \tfrac{2}{3}, \tfrac{1}{3}\right\}; \quad p_j = 0,\ j \notin \{-2,-1,0,1,2\}.$$

Now let $x \in \mathbb{R}$, and denote by k the (unique) integer for which it holds that

$$x \in [k, k+1),$$

that is,

$$x - k \in [0,1),$$

and

$$x - k - 1 \in [-1, 0),$$

whereas

$$x - j \in \mathbb{R} \setminus [-1,1), \quad j \in \mathbb{Z} \setminus \{k, k+1\}.$$

It follows from (2.1.6) that

$$\sum_j h(x-j) = \sum_{j=k}^{k+1} h(x-j) = [1-(x-k)] + [1+(x-k-1)] = 1,$$

from which we deduce that

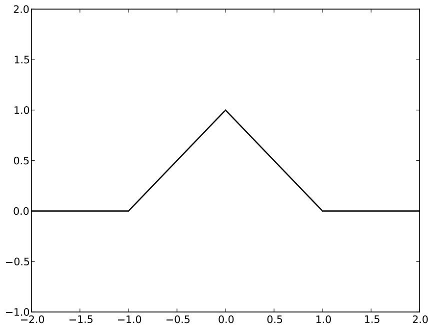
$$\sum_j h(x-j) = 1, \quad x \in \mathbb{R},$$

that is, h provides a partition of unity. Moreover, observe from (2.1.6) that

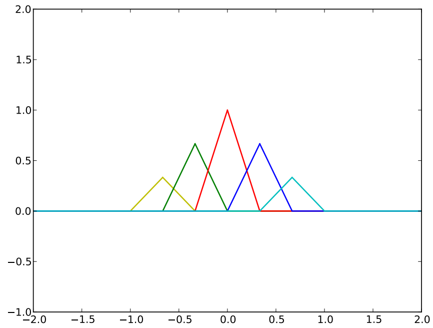
$$\int_{-\infty}^{\infty} h(x)dx = \int_{-1}^{1} h(x)dx = 1.$$

Hence h is a 3-scaling function.

The graph of $h(x)$ is given in Figure 2.2 (a), and the graphs of $\frac{1}{3}h(3x+2)$, $\frac{2}{3}h(3x+1)$, $h(3x)$, $\frac{2}{3}h(3x-1)$ and $\frac{1}{3}h(3x-2)$ are given in Figure 2.2 (b). ■



(a) The hat function h.



(b) Scaled translates of h.

Figure 2.2: Refinability of the hat function h.

Example 2.1.5 Consider the function

$$u(x) := h(x) - h(x-1). \tag{2.1.21}$$

It can be shown as in Example 2.1.4 that u is 3-refinable, with

$$\{p_{-2}, p_{-1}, p_0, p_1, p_2, p_3, p_4\} = \left\{\tfrac{1}{3}, 1, 2, \tfrac{7}{3}, 2, 1, \tfrac{1}{3}\right\}; \qquad p_j = 0,\ j \notin \{-2, -1, 0, 1, 2, 3, 4\},$$

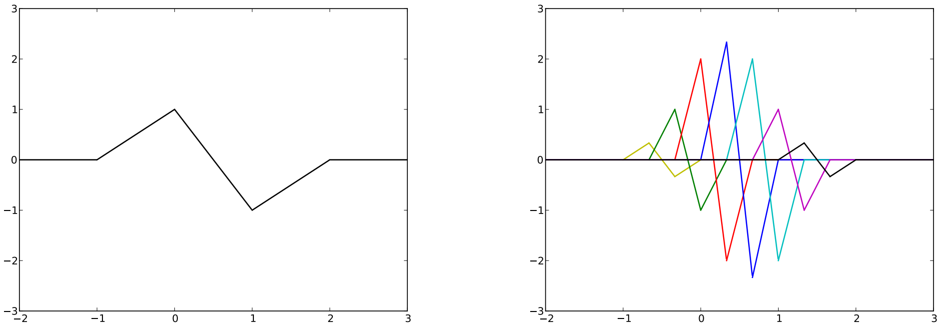
with

$$\sum_j u(x-j) = 0, \quad x \in \mathbb{R},$$

and

$$\int_{-\infty}^{\infty} u(x)dx = \int_{-1}^{2} u(x)dx = 0,$$

so that $\phi = u$ is not a 3-scaling function, since the unit integral condition (2.1.2) is not satisfied.



(a) The function u. (b) Scaled translates of u.

Figure 2.3: Refinability of the function u.

The graph of $u(x)$ is given in Figure 2.3 (a), and the graphs of $\frac{1}{3}u(3x+2)$, $u(3x+1)$, $2u(3x)$, $\frac{7}{3}u(3x-1)$, $2u(3x-2)$, $u(3x-3)$ and $\frac{1}{3}u(3x-4)$ are given in Figure 2.3 (b). ■

Example 2.1.6 As discussed in [7], Example 2.28, pp 34-35, a shifted version of the De Rham function, denoted by ϕ^{DR}, is an interesting example of the 3-refinable function $\phi = \phi^{DR}$, with refinement sequence given by

$$\{p_0, p_1, p_2, p_3, p_4\} = \left\{\tfrac{2}{3}, \tfrac{1}{3}, 1, \tfrac{1}{3}, \tfrac{2}{3}\right\}, \quad p_j = 0, \ j \notin \{0, 1, 2, 3, 4\}.$$

It is shown in [2] that the function ϕ^{DR} is continuous, but nowhere differentiable in its support interval $[0, 2]$, being the limit of a certain fractal process. The graph of ϕ^{DR} can be seen in Figure 2.4. ■

We see that in each of Examples 2.1.4, 2.1.5 and 2.1.6, the length of the support interval of the refinable function is half the length of the support of its corresponding refinement sequence. This is true in general for 3-refinable functions, as we will show in Theorem 2.1.7 below. It is interesting to note that, in contrast, the support interval of a 2-refinable function agrees with the support of its corresponding refinement sequence (see [1], Theorem 2.1.1, p 46).

Theorem 2.1.7 *Let ϕ be a 3-refinable function with refinement sequence $\{p_j\}$, where* $\operatorname{supp}\{p_j\} = [\mu, \nu]\big|_{\mathbb{Z}}$. *Then*

$$\operatorname{supp}^c \phi = \left[\tfrac{1}{2}\mu, \tfrac{1}{2}\nu\right]. \tag{2.1.22}$$

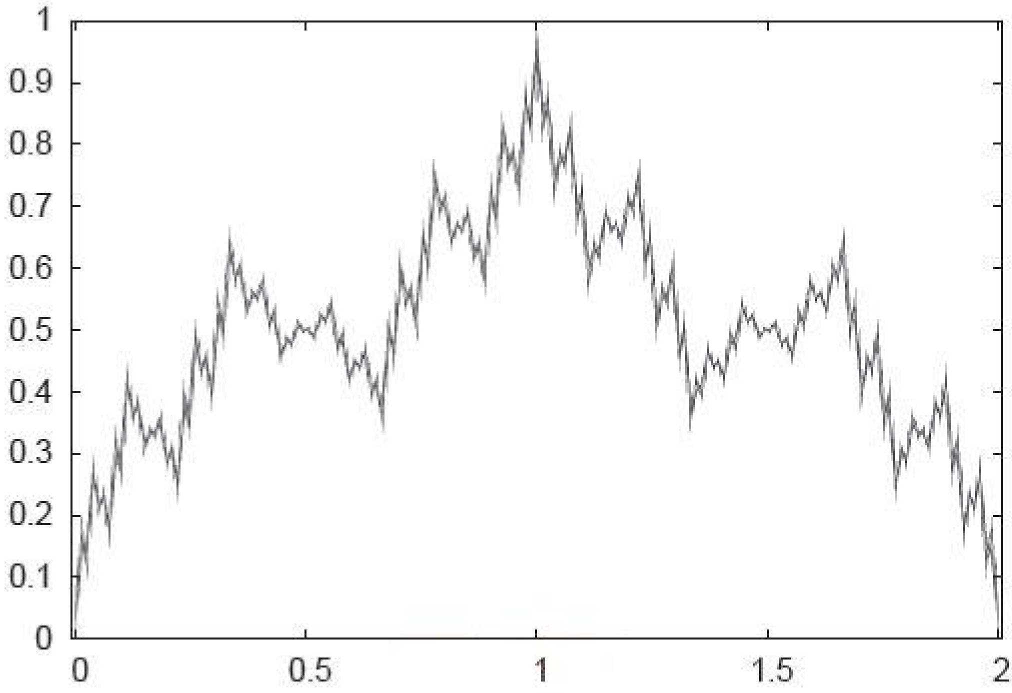


Figure 2.4: The shifted De Rham function ϕ^{DR}.

Proof.

Let

$$m := \inf\{x : \phi(x) \neq 0\}; \quad M := \sup\{x : \phi(x) \neq 0\}.$$

Since $p_\mu \neq 0$ and $p_\nu \neq 0$, and since ϕ is compactly supported, it follows from (2.1.3) that

$$\begin{aligned} m = \inf\{x : \phi(x) \neq 0\} &= \inf\{x : p_\mu \phi(3x - \mu) \neq 0\} \\ &= \inf\{x : \phi(3x - \mu) \neq 0\} = \tfrac{m+\mu}{3}, \end{aligned}$$

and, similarly,

$$\begin{aligned} M = \sup\{x : \phi(x) \neq 0\} &= \sup\{x : p_\nu \phi(3x - \nu) \neq 0\} \\ &= \sup\{x : \phi(3x - \nu) \neq 0\} = \tfrac{M+\nu}{3}. \end{aligned}$$

Hence $m = \frac{1}{2}\mu$ and $M = \frac{1}{2}\nu$, so that

$$\operatorname{supp}^c \phi = [m, M] = \left[\tfrac{1}{2}\mu, \tfrac{1}{2}\nu\right].$$

■

2.2 Ternary subdivision

Having defined basic concepts regarding refinable functions, we proceed to establish a subdivision algorithm for the rendering (or drawing) of the parametric curve

$$\boldsymbol{F_c}(t) = \sum_j \boldsymbol{c}_j \phi(t-j), \quad t \in \mathbb{R}, \tag{2.2.1}$$

where ϕ is a 3-scaling function with refinement sequence $\{p_j\}$, and where the elements of the sequence $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in l(\mathbb{Z})$ are called control points, that is, points selected by the user to determine the shape of the desired curve. The control points satisfy $\boldsymbol{c}_j \in \mathbb{R}^s$, $j \in \mathbb{Z}$, for $s = 2$ or $s = 3$ (as indicated by the bold-faced notation).

In this section, we will present results on ternary subdivision schemes, analogous to the results on binary subdivision schemes, as in Section 3.1 (pp 76-85) and Sections 4.1 to 4.5 (pp 134-163) in [1].

We give a brief explanation of the working of the algorithm. We set $\boldsymbol{c}_j^0 := \boldsymbol{c}_j$. By using the refinement sequence $\{p_j\}$, we then compute the sequences

$$\begin{array}{ccc} \left\{\boldsymbol{c}_j^1\right\} & \text{from} & \left\{\boldsymbol{c}_j^0\right\} \\ \left\{\boldsymbol{c}_j^2\right\} & \text{from} & \left\{\boldsymbol{c}_j^1\right\} \\ & \vdots & \end{array}$$

and thereby increasing the number of points in $\left\{\boldsymbol{c}_j^r\right\}$ and the resolution of $\boldsymbol{F_c}$ at each iteration $r = 1, 2, \ldots$, as will be shown in this section. For r sufficiently large, and for appropriately chosen refinement sequences $\{p_j\}$, the point set $\left\{\boldsymbol{c}_j^r\right\}$ then well represents the curve $\boldsymbol{F_c}$.

This idea is made more precise as follows. First, we use the refinement relation (2.1.3) repeatedly to obtain, from (2.2.1), for any $t \in \mathbb{R}$,

$$\begin{aligned} \boldsymbol{F_c}(t) = \sum_j \boldsymbol{c}_j \left[\sum_k p_k \phi\left(3t-3j-k\right)\right] &= \sum_j \boldsymbol{c}_j \left[\sum_k p_{k-3j} \phi\left(3t-k\right)\right] \\ &= \sum_k \left[\sum_j p_{k-3j} \boldsymbol{c}_j\right] \phi\left(3t-k\right) \\ &= \sum_j \left[\sum_k p_{j-3k} \boldsymbol{c}_k\right] \phi\left(3t-j\right), \end{aligned}$$

and thus

$$\boldsymbol{F_c}(t) = \sum_j \boldsymbol{c}_j^1 \phi(3t-j),$$

where

$$\boldsymbol{c}_j^1 := \sum_k p_{j-3k}\boldsymbol{c}_k, \quad j \in \mathbb{Z}.$$

Continuing this argument, we obtain

$$\boldsymbol{F_c}(t) = \sum_j \boldsymbol{c}_j^r \phi(3^r t - j), \quad t \in \mathbb{R},\ r = 0, 1, \ldots, \tag{2.2.2}$$

where

$$\boldsymbol{c}_j^0 := \boldsymbol{c}_j; \quad \boldsymbol{c}_j^r := \sum_k p_{j-3k}\boldsymbol{c}_k^{r-1}, \quad j \in \mathbb{Z}, \quad r = 1, 2, \ldots. \tag{2.2.3}$$

Note from (2.2.3) that we can rewrite the sequence $\left\{\boldsymbol{c}_j^r\right\}$ in the following way:

$$\left.\begin{aligned} \boldsymbol{c}_{3n}^r &:= \sum_k p_{3n-3k}\boldsymbol{c}_k^{r-1} = \sum_k p_{3k}\boldsymbol{c}_{n-k}^{r-1}, \\ \boldsymbol{c}_{3n-1}^r &:= \sum_k p_{3n-1-3k}\boldsymbol{c}_k^{r-1} = \sum_k p_{3k-1}\boldsymbol{c}_{n-k}^{r-1}, \\ \boldsymbol{c}_{3n-2}^r &:= \sum_k p_{3n-2-3k}\boldsymbol{c}_k^{r-1} = \sum_k p_{3k-2}\boldsymbol{c}_{n-k}^{r-1}, \end{aligned}\right\} \quad n \in \mathbb{Z}, \quad r = 1, 2, \ldots. \tag{2.2.4}$$

We see from (2.2.4) that three ordered point sets or sequences $\{\boldsymbol{c}_{3n}^r\}, \{\boldsymbol{c}_{3n-1}^r\}$ and $\left\{\boldsymbol{c}_{3n-2}^r\right\}$ are generated from $\left\{\boldsymbol{c}_j^{r-1}\right\}$ at each iterative step; we could think of this as shifting the point $\boldsymbol{c}_n^{r-1}$ to a new position $\boldsymbol{c}_{3n}^r$, while two new points, $\boldsymbol{c}_{3n-1}^r$ and $\boldsymbol{c}_{3n-2}^r$, are added at each iteration. This implies that there will be "three times" as many points in $\left\{\boldsymbol{c}_j^r\right\}$ as in $\left\{\boldsymbol{c}_j^{r-1}\right\}$, for each $r = 1, 2, \ldots$, so that the resolution of the desired curve $\boldsymbol{F_c}$ is tripled at each step. Note that when working with 2-refinable functions and their corresponding refinement sequences, there are "twice" as many points in $\left\{\boldsymbol{c}_j^r\right\}$ as in $\left\{\boldsymbol{c}_j^{r-1}\right\}$, for each $r = 1, 2, \ldots$, so that the resolution of $\boldsymbol{F_c}$ is doubled at each iteration, as discussed in [1], p 75.

The process of generating the sequences $\left\{\boldsymbol{c}_j^r\right\}$ from $\left\{\boldsymbol{c}_j^{r-1}\right\}$ (by applying (2.2.4) iteratively) is called a ternary subdivision scheme.

We could also write (2.2.4) in terms of "weight sequences", as follows. Define

$$\left.\begin{aligned} w_j^1 &:= p_{3j}, \\ w_j^2 &:= p_{3j-1}, \\ w_j^3 &:= p_{3j-2}, \end{aligned}\right\} \quad j \in \mathbb{Z}, \tag{2.2.5}$$

according to which (2.2.4) can be written as

$$\left.\begin{aligned} \boldsymbol{c}^r_{3n} &:= \sum_k w^1_k \boldsymbol{c}^{r-1}_{n-k}, \\ \boldsymbol{c}^r_{3n-1} &:= \sum_k w^2_k \boldsymbol{c}^{r-1}_{n-k}, \\ \boldsymbol{c}^r_{3n-2} &:= \sum_k w^3_k \boldsymbol{c}^{r-1}_{n-k}, \end{aligned}\right\} \quad n \in \mathbb{Z}, \quad r = 1, 2, \ldots. \tag{2.2.6}$$

It is natural to require that the weight sequences sum to one. We define this to be the "sum-rule" property of the refinement sequence $\{p_j\}$.

Definition 2.2.1 *A refinement sequence $\{p_j\}$ is said to possess the sum-rule property (or to satisfy the sum-rule condition) if*

$$\sum_j w^1_j = 1; \quad \sum_j w^2_j = 1; \quad \sum_j w^3_j = 1, \tag{2.2.7}$$

that is,

$$\sum_j p_{3j} = 1; \quad \sum_j p_{3j-1} = 1; \quad \sum_j p_{3j-2} = 1. \tag{2.2.8}$$

In practice, it is often necessary to require that the limit curve pass through the initial control points, that is, the original sequence of control points $\{\boldsymbol{c}_j\} = \{\boldsymbol{c}^0_j\}$ are kept fixed at each iteration, while we add two new points at each iterative step. This process is called ternary interpolatory subdivision, and is defined by replacing the first line of (2.2.4) with the interpolatory condition

$$\boldsymbol{c}^r_{3n} := \boldsymbol{c}^{r-1}_n, \quad n \in \mathbb{Z}, \quad r = 1, 2, \ldots, \tag{2.2.9}$$

so that (2.2.4) becomes

$$\left.\begin{aligned} \boldsymbol{c}^r_{3n} &:= \boldsymbol{c}^{r-1}_n, \\ \boldsymbol{c}^r_{3n-1} &:= \sum_k p_{3n-1-3k} \boldsymbol{c}^{r-1}_k = \sum_k p_{3k-1} \boldsymbol{c}^{r-1}_{n-k}, \\ \boldsymbol{c}^r_{3n-2} &:= \sum_k p_{3n-2-3k} \boldsymbol{c}^{r-1}_k = \sum_k p_{3k-2} \boldsymbol{c}^{r-1}_{n-k}, \end{aligned}\right\} \quad n \in \mathbb{Z}, \quad r = 1, 2, \ldots, \tag{2.2.10}$$

or, in terms of weight sequences,

$$\left.\begin{aligned} \boldsymbol{c}^r_{3n} &:= \boldsymbol{c}^{r-1}_n, \\ \boldsymbol{c}^r_{3n-1} &:= \sum_k w^2_k \boldsymbol{c}^{r-1}_{n-k}, \\ \boldsymbol{c}^r_{3n-2} &:= \sum_k w^3_k \boldsymbol{c}^{r-1}_{n-k}, \end{aligned}\right\} \quad n \in \mathbb{Z}, \quad r = 1, 2, \ldots. \tag{2.2.11}$$

We proceed to define the concept of an interpolatory refinable function.

Definition 2.2.2 *A refinable function ϕ is called an interpolatory refinable function if it satisfies the canonical interpolation property*

$$\phi(j) = \delta_j, \quad j \in \mathbb{Z}, \tag{2.2.12}$$

with $\{\delta_j\}$ denoting the Kronecker delta sequence, that is,

$$\delta_j := \begin{cases} 1, & j = 0; \\ 0, & j \in \mathbb{Z} \setminus \{0\}. \end{cases} \tag{2.2.13}$$

Note that, when applying interpolatory subdivision, we require that the limit curve contains the initial control points, with, specifically,

$$\boldsymbol{F_c}(j) = \boldsymbol{c}_j, \quad j \in \mathbb{Z},$$

that is, from (2.2.1),

$$\sum_k \boldsymbol{c}_k \phi(j-k) = \boldsymbol{c}_j, \quad j \in \mathbb{Z},$$

which is satisfied if ϕ satisfies the canonical interpolation property (2.2.12).

For more convenient notation, we now introduce the concept of a subdivision operator.

Definition 2.2.3 *Let $\boldsymbol{p} = \{p_j\} \in l_0$. We define the subdivision operator $\mathcal{S}_{\boldsymbol{p}}$ corresponding to $\boldsymbol{p}$ by*

$$(\mathcal{S}_{\boldsymbol{p}}\boldsymbol{c})_j := \sum_k p_{j-3k}\boldsymbol{c}_k, \quad j \in \mathbb{Z}, \tag{2.2.14}$$

for any $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in l(\mathbb{Z})$.

Using this definition, we note that (2.2.3) can be reformulated in terms of the subdivision operator $\mathcal{S}_{\boldsymbol{p}}$ as

$$\boldsymbol{c}_j^0 := \boldsymbol{c}_j; \quad \boldsymbol{c}_j^r := (\mathcal{S}_{\boldsymbol{p}}\boldsymbol{c}^{r-1})_j = (\mathcal{S}_{\boldsymbol{p}}^r\boldsymbol{c})_j, \quad j \in \mathbb{Z}, \quad r = 1, 2, \ldots, \tag{2.2.15}$$

where $\mathcal{S}_{\boldsymbol{p}}^r := \mathcal{S}_{\boldsymbol{p}}\mathcal{S}_{\boldsymbol{p}}^{r-1}$, with $\mathcal{S}_{\boldsymbol{p}}^0$ denoting the identity operator on $l(\mathbb{Z})$, and $\mathcal{S}_{\boldsymbol{p}}^1 = \mathcal{S}_{\boldsymbol{p}}$.

We also have the following definition for an interpolatory subdivision operator.

Definition 2.2.4 *A subdivision operator $\mathcal{S}_{\boldsymbol{p}}$ corresponding to some sequence $\boldsymbol{p} = \{p_j\} \in l_0$ is called an interpolatory subdivision operator if it satisfies the condition*

$$(\mathcal{S}_{\boldsymbol{p}}\boldsymbol{c})_{3j} = \boldsymbol{c}_j, \quad j \in \mathbb{Z}, \tag{2.2.16}$$

for any $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in l(\mathbb{Z})$.

We end this section by defining the term "convergent subdivision scheme", as follows.

Definition 2.2.5 *$\mathcal{S}_{\boldsymbol{p}}$ is said to provide a convergent subdivision scheme if there exists a non-trivial function $\phi_{\boldsymbol{p}} \in C(\mathbb{R})$ such that*

$$E_{\boldsymbol{p}}(r) := \sup_j \left|\phi_{\boldsymbol{p}}\left(\frac{j}{3^r}\right) - p_j^{[r]}\right| \to 0, \quad r \to \infty, \tag{2.2.17}$$

where

$$p_j^{[r]} := (\mathcal{S}_{\boldsymbol{p}}^r\boldsymbol{\delta})_j, \quad j \in \mathbb{Z}, \ r = 1, 2, \ldots, \tag{2.2.18}$$

with $\boldsymbol{\delta} = \{\delta_j\}$ denoting the Kronecker delta sequence, as in (2.2.13). We call $\phi_{\boldsymbol{p}}$ the limit function corresponding to $\mathcal{S}_{\boldsymbol{p}}$.

In the two following sections, we will devote our attention to studying subdivision operators providing convergent subdivision schemes, and the corresponding limit functions.

2.3 Subdivision convergence

We start by deriving a necessary condition for subdivision convergence, analogous to Theorem 4.1.1, p 136, in [1] for binary subdivision convergence. The proof here is a straightforward adaptation of the proof in [1].

Theorem 2.3.1 *Let $\boldsymbol{p} = \{p_j\} \in l_0$ with $\operatorname{supp}\{p_j\} = [\mu, \nu]\big|_{\mathbb{Z}}$, and suppose that the corresponding subdivision operator $\mathcal{S}_{\boldsymbol{p}}$ provides a convergent subdivision scheme. Then $\{p_j\}$ must satisfy the sum-rule condition (2.2.8).*

Proof.
Let $\phi_{\boldsymbol{p}}$ denote the limit function corresponding to $\mathcal{S}_{\boldsymbol{p}}$. Since $\phi_{\boldsymbol{p}}$ is non-trivial, there exists a point $x \in \mathbb{R}$ such that $\phi_{\boldsymbol{p}}(x) \neq 0$. For this x, let $\{j_r : r = 0, 1, \ldots\} \subseteq \mathbb{Z}$ denote a sequence such that

$$\left|x - \frac{j_r}{3^r}\right| \to 0, \quad r \to \infty. \tag{2.3.1}$$

Let $l \in \{0, 1, 2\}$ be a fixed integer, and define the sequence

$$j_{l,r} := 3\lfloor \tfrac{j_r}{3} \rfloor + l, \quad r = 0, 1, \ldots, \tag{2.3.2}$$

for which it follows, for any $r = 0, 1, \ldots,$ that

$$\begin{aligned} x - \tfrac{j_{l,r}}{3^r} &= x - \tfrac{1}{3^r}\left(3\lfloor \tfrac{j_r}{3} \rfloor + l\right) \\ &\geq x - \tfrac{1}{3^r}\left(3(\tfrac{j_r}{3}) + l\right) \\ &= \left(x - \tfrac{j_r}{3^r}\right) - \tfrac{l}{3^r}, \end{aligned}$$

whereas, since $\lceil x \rceil - 1 \leq \lfloor x \rfloor$,

$$\begin{aligned} x - \tfrac{j_{l,r}}{3^r} &= x - \tfrac{1}{3^r}\left(3\lfloor \tfrac{j_r}{3} \rfloor + l\right) \\ &\leq x - \tfrac{1}{3^r}\left(3\left(\lceil \tfrac{j_r}{3} \rceil - 1\right) + l\right) \\ &\leq x - \tfrac{1}{3^r}\left(3\left(\tfrac{j_r}{3} - 1\right) + l\right) \\ &= x - \tfrac{j_r}{3^r} + \tfrac{3}{3^r} - \tfrac{l}{3^r} \\ &= \left(x - \tfrac{j_r}{3^r}\right) - \tfrac{l-3}{3^r}, \end{aligned}$$

yielding

$$\left(x - \tfrac{j_r}{3^r}\right) - \tfrac{l}{3^r} \leq x - \tfrac{j_{l,r}}{3^r} \leq \left(x - \tfrac{j_r}{3^r}\right) - \tfrac{l-3}{3^r}.$$

Hence, since $|x - \frac{j_r}{3^r}| \to 0, \;\; r \to \infty$, we have

$$|x - \tfrac{j_{l,r}}{3^r}| \to 0, \quad r \to \infty.$$

By the continuity of $\phi_{\boldsymbol{p}}$, it then follows that

$$\left| \phi_{\boldsymbol{p}}(x) - \phi_{\boldsymbol{p}}\left(\tfrac{j_{l,r}}{3^r}\right) \right| \to 0, \quad r \to \infty. \tag{2.3.3}$$

Also, since $\mathcal{S}_{\boldsymbol{p}}$ provides a convergent subdivision scheme, we have, from (2.2.17),

$$\left| \phi_{\boldsymbol{p}}\left(\tfrac{j_{l,r}}{3^r}\right) - p^{[r]}_{j_{l,r}} \right| \to 0, \quad r \to \infty. \tag{2.3.4}$$

Combining (2.3.3) and (2.3.4), we obtain

$$\begin{aligned} |\phi_{\boldsymbol{p}}(x) - p^{[r]}_{j_{l,r}}| &= \left| \phi_{\boldsymbol{p}}(x) - \phi_{\boldsymbol{p}}\left(\tfrac{j_{l,r}}{3^r}\right) + \phi_{\boldsymbol{p}}\left(\tfrac{j_{l,r}}{3^r}\right) - p^{[r]}_{j_{l,r}} \right| \\ &\leq \left| \phi_{\boldsymbol{p}}(x) - \phi_{\boldsymbol{p}}\left(\tfrac{j_{l,r}}{3^r}\right) \right| + \left| \phi_{\boldsymbol{p}}\left(\tfrac{j_{l,r}}{3^r}\right) - p^{[r]}_{j_{l,r}} \right| \\ &\to 0, \quad r \to \infty. \end{aligned} \tag{2.3.5}$$

Now let $r \in \mathbb{N}$ be fixed. It follows from (2.2.18) and (2.2.14) that

$$\begin{aligned} \phi_{\boldsymbol{p}}(x) - p^{[r]}_{j_{l,r}} &= \phi_{\boldsymbol{p}}(x) - \mathcal{S}_{\boldsymbol{p}}\left(\mathcal{S}^{r-1}_{\boldsymbol{p}}\boldsymbol{\delta}\right)_{j_{l,r}} \\ &= \phi_{\boldsymbol{p}}(x) - \sum_k p_{j_{l,r}-3k}\left(\mathcal{S}^{r-1}_{\boldsymbol{p}}\boldsymbol{\delta}\right)_k \\ &= \phi_{\boldsymbol{p}}(x) - \sum_k p_{j_{l,r}-3k} p^{[r-1]}_k \end{aligned}$$

$$= \phi_{\boldsymbol{p}}(x)\left[1 - \sum_k p_{j_{l,r}-3k}\right] + \sum_k p_{j_{l,r}-3k}\left[\phi_{\boldsymbol{p}}(x) - p_k^{[r-1]}\right],$$

so that, by applying also $\operatorname{supp}\{p_j\} = [\mu,\nu]\big|_{\mathbb{Z}}$, we obtain

$$\begin{aligned}\left|\phi_{\boldsymbol{p}}(x)\right|\left|1 - \sum_k p_{j_{l,r}-3k}\right| &= \left|\phi_{\boldsymbol{p}}(x) - p_{j_{l,r}}^{[r]} - \sum_k p_{j_{l,r}-3k}\left(\phi_{\boldsymbol{p}}(x) - p_k^{[r-1]}\right)\right| \\ &\leq \left|\phi_{\boldsymbol{p}}(x) - p_{j_{l,r}}^{[r]}\right| + \sum_{k=\mu_{l,r}}^{\nu_{l,r}}\left|p_{j_{l,r}-3k}\right|\left|\phi_{\boldsymbol{p}}(x) - p_k^{[r-1]}\right|,\end{aligned}$$

with

$$\mu_{l,r} := \lceil \tfrac{1}{3}(j_{l,r} - \nu)\rceil; \quad \nu_{l,r} := \lfloor \tfrac{1}{3}(j_{l,r} - \mu)\rfloor, \tag{2.3.6}$$

and thus

$$\nu_{l,r} - \mu_{l,r} \leq \tfrac{1}{3}(j_{l,r} - \mu) - \tfrac{1}{3}(j_{l,r} - \nu) = \tfrac{\nu-\mu}{3}.$$

Hence it follows that

$$\begin{aligned}&\left|\phi_{\boldsymbol{p}}(x)\right|\left|1 - \sum_k p_{j_{l,r}-3k}\right| \\ &\leq \left|\phi_{\boldsymbol{p}}(x) - p_{j_{l,r}}^{[r]}\right| + \left[\max_{\mu\leq j\leq\nu}|p_j|\right]\left[\max_{\mu_{l,r}\leq k\leq\nu_{l,r}}\left|\phi_{\boldsymbol{p}}(x) - p_k^{[r-1]}\right|\right]\left(\tfrac{\nu-\mu}{3} + 1\right).\end{aligned} \tag{2.3.7}$$

Now let $k_{l,r}$ denote some integer in the set $\{\mu_{l,r},\ldots,\nu_{l,r}\}$ for which

$$\left|\phi_{\boldsymbol{p}}(x) - p_{k_{l,r}}^{[r-1]}\right| = \max_{\mu_{l,r}\leq k\leq\nu_{l,r}}\left|\phi_{\boldsymbol{p}}(x) - p_k^{[r-1]}\right|. \tag{2.3.8}$$

Then we have, from (2.3.6),

$$\begin{aligned}x - \tfrac{k_{l,r}}{3^{r-1}} &\leq x - \tfrac{\mu_{l,r}}{3^{r-1}} \\ &= x - \tfrac{1}{3^{r-1}}\left(\lceil \tfrac{1}{3}(j_{l,r} - \nu)\rceil\right) \\ &\leq x - \tfrac{1}{3^{r-1}}\left(\tfrac{1}{3}(j_{l,r} - \nu)\right) \\ &= \left(x - \tfrac{j_{l,r}}{3^r}\right) + \tfrac{\nu}{3^r},\end{aligned}$$

whereas

$$\begin{aligned}x - \tfrac{k_{l,r}}{3^{r-1}} &\geq x - \tfrac{\nu_{l,r}}{3^{r-1}} \\ &= x - \tfrac{1}{3^{r-1}}\left(\lfloor \tfrac{1}{3}(j_{l,r} - \mu)\rfloor\right) \\ &\geq x - \tfrac{1}{3^{r-1}}\left(\tfrac{1}{3}(j_{l,r} - \mu)\right) \\ &= \left(x - \tfrac{j_{l,r}}{3^r}\right) + \tfrac{\mu}{3^r},\end{aligned}$$

yielding

$$\left(x-\tfrac{j_{l,r}}{3^r}\right)+\tfrac{\mu}{3^r}\leq x-\tfrac{k_{l,r}}{3^{r-1}}\leq\left(x-\tfrac{j_{l,r}}{3^r}\right)+\tfrac{\nu}{3^r}.$$

Hence, since $|x-\frac{j_{l,r}}{3^r}|\to 0,\ \ r\to\infty$, we have

$$|x-\tfrac{k_{l,r}}{3^{r-1}}|\to 0,\quad r\to\infty.$$

By the continuity of $\phi_{\boldsymbol{p}}$, it then follows that

$$\left|\phi_{\boldsymbol{p}}(x)-\phi_{\boldsymbol{p}}\left(\tfrac{k_{l,r}}{3^{r-1}}\right)\right|\to 0,\quad r\to\infty. \tag{2.3.9}$$

Also, since $\mathcal{S}_{\boldsymbol{p}}$ provides a convergent subdivision scheme, we have, from (2.2.17),

$$\left|\phi_{\boldsymbol{p}}\left(\tfrac{k_{l,r}}{3^{r-1}}\right)-p^{[r-1]}_{k_{l,r}}\right|\to 0,\quad r\to\infty. \tag{2.3.10}$$

Combining (2.3.9) and (2.3.10), we obtain

$$\begin{aligned}|\phi_{\boldsymbol{p}}(x)-p^{[r-1]}_{k_{l,r}}| &= \left|\phi_{\boldsymbol{p}}(x)-\phi_{\boldsymbol{p}}\left(\tfrac{k_{l,r}}{3^{r-1}}\right)+\phi_{\boldsymbol{p}}\left(\tfrac{k_{l,r}}{3^{r-1}}\right)-p^{[r-1]}_{k_{l,r}}\right|\\ &\leq\left|\phi_{\boldsymbol{p}}(x)-\phi_{\boldsymbol{p}}\left(\tfrac{k_{l,r}}{3^{r-1}}\right)\right|+\left|\phi_{\boldsymbol{p}}\left(\tfrac{k_{l,r}}{3^{r-1}}\right)-p^{[r-1]}_{k_{l,r}}\right|\\ &\to 0,\quad r\to\infty. \end{aligned}\tag{2.3.11}$$

Hence, by using (2.3.8), (2.3.5) and (2.3.11) in (2.3.7), we have

$$\begin{aligned}&\left|\phi_{\boldsymbol{p}}(x)\right|\left|1-\sum_k p_{j_{l,r}-3k}\right|\\ &\leq\left|\phi_{\boldsymbol{p}}(x)-p^{[r]}_{j_{l,r}}\right|+\left[\max_{\mu\leq j\leq\nu}|p_j|\right]\left[\left|\phi_{\boldsymbol{p}}(x)-p^{[r-1]}_{k_{l,r}}\right|\right]\left(\tfrac{\nu-\mu}{3}+1\right)\\ &\to 0+0=0,\quad r\to\infty.\end{aligned}\tag{2.3.12}$$

Finally, we note from (2.3.2) that

$$\sum_k p_{j_{l,r}-3k}=\sum_k p_{l-3k},\quad r=0,1,\ldots. \tag{2.3.13}$$

Hence, from (2.3.12), together with the fact that $\phi_{\boldsymbol{p}}(x)\neq 0$, we deduce that

$$\sum_k p_{l-3k}=1,\quad l=0,1,2, \tag{2.3.14}$$

which is equivalent to the sum-rule property (2.2.8). ■

We will rely on the following properties of the sequence $\left\{p^{[r]}_j\right\}$. The analogous result for binary subdivison can be found in [1] (Theorem 4.2.1, pp 138-139). The proof below follows the same pattern as the proof in [1], although the result in (ii) is a non-trivial extension of the analogous result in [1].

Theorem 2.3.2 *Let $\boldsymbol{p} = \{p_j\} \in l_0$, with $\operatorname{supp}\{p_j\} = [\mu,\nu]\big|_{\mathbb{Z}}$, and let the sequences $\left\{p_j^{[r]} : j \in \mathbb{Z}\right\}$, $r \in \mathbb{N}$, be defined by (2.2.18). Then the following hold:*

(i) The sequences $\left\{p_j^{[r]}\right\}$ satisfy the recursion formulation

$$p_j^{[1]} = p_j; \quad p_j^{[r]} = \sum_k p_k p_{j-3^{r-1}k}^{[r-1]}, \quad r = 2,3,\ldots, \quad j \in \mathbb{Z}; \tag{2.3.15}$$

(ii) The sequences $\left\{p_j^{[r]}\right\}$ are finitely supported, with

$$\operatorname{supp}\left\{p_j^{[r]}\right\} = \left[\left(\tfrac{3^r-1}{2}\right)\mu, \left(\tfrac{3^r-1}{2}\right)\nu\right]\Big|_{\mathbb{Z}}, \quad r \in \mathbb{N}; \tag{2.3.16}$$

(iii) If $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in l(\mathbb{Z})$ is any sequence of control points, the subdivision scheme (2.2.15) can be reformulated as

$$\left(\mathcal{S}_{\boldsymbol{p}}^r \boldsymbol{c}\right)_j = \sum_k p_{j-3^r k}^{[r]} \boldsymbol{c}_k, \quad j \in \mathbb{Z}, \quad r \in \mathbb{N}; \tag{2.3.17}$$

(iv) If $\{p_j\}$ satisfies the sum-rule property (2.2.8), then $\left\{p_j^{[r]}\right\}$ satisfies the condition

$$\sum_k p_{j-3^r k}^{[r]} = 1, \quad j \in \mathbb{Z}, \quad r \in \mathbb{N}. \tag{2.3.18}$$

Proof.

(i) Let $r = 1$. Then, from (2.2.18), (2.2.14) and (2.2.13), we have

$$p_j^{[1]} = \left(\mathcal{S}_{\boldsymbol{p}}\boldsymbol{\delta}\right)_j = \sum_k p_{j-3k}\delta_k = p_j, \tag{2.3.19}$$

which proves the first equation in (2.3.15). Now let $r \in \{2,3,\ldots\}$. Our proof is by induction on r. If $r = 2$, we have, from (2.3.19), (2.2.14) and (2.2.18),

$$\begin{aligned}
\sum_k p_k p_{j-3k}^1 = \sum_k p_{j-3k} p_k &= \left(\mathcal{S}_{\boldsymbol{p}}\boldsymbol{p}\right)_j \\
&= \left(\mathcal{S}_{\boldsymbol{p}}\left(\mathcal{S}_{\boldsymbol{p}}\boldsymbol{\delta}\right)\right)_j \\
&= \left(\mathcal{S}_{\boldsymbol{p}}^2\boldsymbol{\delta}\right)_j \\
&= p_j^{[2]},
\end{aligned}$$

so that the second equation in (2.3.15) holds for $r = 2$. Now assume the result holds for some $r \geq 3$. It then follows from repeated applications of (2.2.18) and (2.2.14), together with the inductive assumption, that

$$\begin{aligned}
p_j^{[r+1]} &= \left(\mathcal{S}_{\boldsymbol{p}}^{r+1}\boldsymbol{\delta}\right)_j \\
&= \left(\mathcal{S}_{\boldsymbol{p}}\left(\mathcal{S}_{\boldsymbol{p}}^{r}\boldsymbol{\delta}\right)\right)_j \\
&= \sum_k p_{j-3k}\left(\mathcal{S}_{\boldsymbol{p}}^{r}\boldsymbol{\delta}\right)_k \\
&= \sum_k p_{j-3k}\left[\sum_l p_l\left(\mathcal{S}_{\boldsymbol{p}}^{r-1}\boldsymbol{\delta}\right)_{k-3^{r-1}l}\right] \\
&= \sum_l p_l\left[\sum_k p_{j-3k}\left(\mathcal{S}_{\boldsymbol{p}}^{r-1}\boldsymbol{\delta}\right)_{k-3^{r-1}l}\right] \\
&= \sum_l p_l\left[\sum_k p_{j-3k-3^r l}\left(\mathcal{S}_{\boldsymbol{p}}^{r-1}\boldsymbol{\delta}\right)_k\right] \\
&= \sum_l p_l\left(\mathcal{S}_{\boldsymbol{p}}\left(\mathcal{S}_{\boldsymbol{p}}^{r-1}\boldsymbol{\delta}\right)\right)_{j-3^r l} \\
&= \sum_l p_l\left(\mathcal{S}_{\boldsymbol{p}}^{r}\boldsymbol{\delta}\right)_{j-3^r l} \\
&= \sum_l p_l p_{j-3^r l}^{[r]},
\end{aligned}$$

completing our inductive proof of the recursion formulation (2.3.15).

(ii) From (2.3.15), and using the fact that $\operatorname{supp}\{p_j\} = [\mu,\nu]\big|_{\mathbb{Z}}$, we have, for $r = 2, 3, \ldots,$ and any $j \in \mathbb{Z}$,

$$p_j^{[r]} = \sum_{k=\mu}^{\nu} p_k p_{j-3^{r-1}k}^{[r-1]}. \tag{2.3.20}$$

Now let $k \in \{\mu, \ldots, \nu\}$. We note that, for $j < \left(\frac{3^{r+1}-1}{2}\right)\mu$, we have

$$j - 3^r k < \left(\frac{3^{r+1}-1}{2}\right)\mu - 3^r\mu = \frac{3^r}{2}(3\mu - 2\mu) - \frac{1}{2}\mu = \left(\frac{3^r-1}{2}\right)\mu,$$

whereas, for $j > \left(\frac{3^{r+1}-1}{2}\right)\nu$, we have

$$j - 3^r k > \left(\frac{3^{r+1}-1}{2}\right)\nu - 3^r\nu = \frac{3^r}{2}(3\nu - 2\nu) - \frac{1}{2}\nu = \left(\frac{3^r-1}{2}\right)\nu,$$

so that, for $k \in \{\mu, \ldots, \nu\}$, we have

$$j \notin \left\{\left(\frac{3^{r+1}-1}{2}\right)\mu, \ldots, \left(\frac{3^{r+1}-1}{2}\right)\nu\right\} \Rightarrow j - 3^r k \notin \left\{\left(\frac{3^r-1}{2}\right)\mu, \ldots, \left(\frac{3^r-1}{2}\right)\nu\right\}. \tag{2.3.21}$$

We now proceed to prove (2.3.16) by induction on r. If $r = 1$, we have, from (2.3.15) and the fact that $\text{supp}\{p_j\} = [\mu, \nu]\big|_{\mathbb{Z}}$,

$$p_j^{[1]} = p_j = 0, \quad j \notin \{\mu, \ldots, \nu\},$$

so that (2.3.16) holds for $r = 1$. Assume the result holds for some $r \geq 2$. It follows from (2.3.20), (2.3.21) and the inductive assumption that

$$p_j^{[r+1]} = \sum_{k=\mu}^{\nu} p_k p_{j-3^r k}^{[r]} = 0, \quad j \notin \left\{ \left(\tfrac{3^{r+1}-1}{2} \right) \mu, \ldots, \left(\tfrac{3^{r+1}-1}{2} \right) \nu \right\},$$

and thereby completing our inductive proof of (2.3.16).

(iii) Let $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in l(\mathbb{Z})$. Our proof is by induction on r. The result holds for $r = 1$, since, from (2.2.14) and (2.3.15),

$$(\mathcal{S}_{\boldsymbol{p}} \boldsymbol{c})_j = \sum_k p_{j-3k} \boldsymbol{c}_k = \sum_k p_{j-3k}^{[1]} \boldsymbol{c}_k.$$

Assume the result holds for some integer $r \geq 2$. It then follows from (2.3.15) and (2.2.14), together with the inductive assumption, that

$$\begin{aligned}
\sum_k p_{j-3^{r+1}k}^{[r+1]} \boldsymbol{c}_k &= \sum_k \left[\sum_l p_l p_{j-3^{r+1}k-3^r l}^{[r]} \right] \boldsymbol{c}_k \\
&= \sum_k \left[\sum_l p_l p_{j-3^r(3k+l)}^{[r]} \right] \boldsymbol{c}_k \\
&= \sum_k \left[\sum_l p_{l-3k} p_{j-3^r l}^{[r]} \right] \boldsymbol{c}_k \\
&= \sum_l p_{j-3^r l}^{[r]} \left[\sum_k p_{l-3k} \boldsymbol{c}_k \right] \\
&= \sum_l p_{j-3^r l}^{[r]} (\mathcal{S}_{\boldsymbol{p}} \boldsymbol{c})_l \\
&= \left(\mathcal{S}_{\boldsymbol{p}}^r (\mathcal{S}_{\boldsymbol{p}} \boldsymbol{c}) \right)_j = \left(\mathcal{S}_{\boldsymbol{p}}^{r+1} \boldsymbol{c} \right)_j,
\end{aligned}$$

which completes our proof.

(iv) Suppose $\{p_j\}$ satisfies the sum-rule condition (2.2.8), which has the equivalent formulation

$$\sum_k p_{j-3k} = 1, \quad j \in \mathbb{Z}. \tag{2.3.22}$$

Our proof is by induction on r. The result holds for $r = 1$, since, for any $j \in \mathbb{Z}$,

$$\sum_k p^{[1]}_{j-3k} = \sum_k p_{j-3k} = 1,$$

from (2.3.15) and (2.3.22). Assume (2.3.18) holds for some $r \geq 2$. By using (2.3.15) and (2.3.22), together with the inductive assumption, we obtain

$$\begin{aligned}\sum_k p^{[r+1]}_{j-3^{r+1}k} = \sum_k \left[\sum_l p_l p^{[r]}_{j-3^{r+1}k-3^r l}\right] &= \sum_k \left[\sum_l p_l p^{[r]}_{j-3^r(3k+l)}\right] \\ &= \sum_k \left[\sum_l p_{l-3k} p^{[r]}_{j-3^r l}\right] \\ &= \sum_l p^{[r]}_{j-3^r l} \left[\sum_k p_{l-3k}\right] \\ &= \sum_l p^{[r]}_{j-3^r l}(1) = 1,\end{aligned}$$

and thereby completing our inductive proof. ■

2.4 The limit function

This section will be devoted to obtaining properties of the limit function of a convergent subdivision scheme, analogous to properties presented in Sections 4.3 to 4.5, pp 141-163, in [1].

Definition 2.4.1 *Let $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in l(\mathbb{Z})$ be a sequence of control points in $\mathbb{R}^s$, with $s = 2$ or 3. Then the backward difference operator Δ is defined by*

$$(\Delta \boldsymbol{c})_j := \boldsymbol{c}_j - \boldsymbol{c}_{j-1}, \quad j \in \mathbb{Z}. \tag{2.4.1}$$

Also, for any $k \in \mathbb{N}$, the k^{th} order backward difference operator Δ^k is defined by

$$(\Delta^0 \boldsymbol{c})_j := \boldsymbol{c}_j; \quad (\Delta^k \boldsymbol{c})_j := \left(\Delta\left(\Delta^{k-1}\boldsymbol{c}\right)\right)_j. \tag{2.4.2}$$

Observe in particular from (2.4.1) and (2.4.2) that

$$(\Delta^2 \boldsymbol{c})_j = \boldsymbol{c}_j - 2\boldsymbol{c}_{j-1} + \boldsymbol{c}_{j-2}, \quad j \in \mathbb{Z}. \tag{2.4.3}$$

We are now ready to prove the following properties of the limit function of a convergent subdivision scheme. The proof follows the same pattern as the proof of the analogous result for binary subdivision schemes (see Theorem

4.3.1, p 142, [1]).

Note that, for $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in l(\mathbb{Z})$, with $\boldsymbol{c}_j \in \mathbb{R}^s, j \in \mathbb{Z}, s = 2, 3$, we define

$$|\boldsymbol{c}_j| = |(c_{j,1}, \ldots, c_{j,s})| := \sqrt{c_{j,1}^2 + \cdots + c_{j,s}^2}. \tag{2.4.4}$$

Theorem 2.4.2 *Let $\boldsymbol{p} = \{p_j\} \in l_0$, with $\operatorname{supp}\{p_j\} = [\mu, \nu]\big|_{\mathbb{Z}}$. Also, suppose that the corresponding subdivision operator $\mathcal{S}_{\boldsymbol{p}}$ provides a convergent subdivision scheme, and let $\phi_{\boldsymbol{p}}$ denote the corresponding limit function. Then we have the following:*

(i)

$$\phi_{\boldsymbol{p}}(x) = 0, \quad x \notin \left(\tfrac{\mu}{2}, \tfrac{\nu}{2}\right); \tag{2.4.5}$$

(ii) $\phi_{\boldsymbol{p}}$ is refinable with refinement sequence $\{p_j\}$, that is,

$$\phi_{\boldsymbol{p}}(x) = \sum_j p_j \phi_{\boldsymbol{p}}(3x - j), \quad x \in \mathbb{R}; \tag{2.4.6}$$

(iii) $\phi_{\boldsymbol{p}}$ provides a partition of unity, that is,

$$\sum_j \phi_{\boldsymbol{p}}(x - j) = 1, \quad x \in \mathbb{R}; \tag{2.4.7}$$

(iv) For any non-zero control point sequence $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in l(\mathbb{Z})$, with $\Delta \boldsymbol{c} \in l^\infty$, the subdivision scheme (2.2.15) converges uniformly in the sense that

$$\sup_j \left| \boldsymbol{F}_{\boldsymbol{c}}\left(\tfrac{j}{3^r}\right) - \boldsymbol{c}_j^r \right| = \sup_j \left| \sum_k \boldsymbol{c}_k \phi_{\boldsymbol{p}}\left(\tfrac{j}{3^r} - k\right) - \boldsymbol{c}_j^r \right| \to 0, \quad r \to \infty. \tag{2.4.8}$$

Proof.

Throughout this proof, if $x \in \mathbb{R}$, then $\{j_r : r \in \mathbb{N}\}$ will denote a sequence in $\mathbb{Z}$ such that

$$\tfrac{j_r}{3^r} \to x, \quad r \to \infty, \tag{2.4.9}$$

so that, since $\mathcal{S}_{\boldsymbol{p}}$ provides a convergent subdivision scheme, we have

$$\left| \phi_{\boldsymbol{p}}\left(\frac{j_r}{3^r}\right) - p_{j_r}^{[r]} \right| \to 0, \quad r \to \infty, \tag{2.4.10}$$

from (2.2.17), with $\left\{p_j^{[r]}\right\}$ defined, for $r \in \mathbb{N}$, by (2.2.18).

(i) Let $x \in \mathbb{R}$, such that $x \notin \left[\frac{\mu}{2}, \frac{\nu}{2}\right]$. By using (2.4.9), we deduce the existence of an integer $r_0 \in \mathbb{N}$ such that

$$\tfrac{j_r}{3^r} \notin \left[\left(1 - \tfrac{1}{3^r}\right)\left(\tfrac{\mu}{2}\right), \left(1 - \tfrac{1}{3^r}\right)\left(\tfrac{\nu}{2}\right)\right]\Big|_{\mathbb{Z}}, \quad r \geq r_0,$$

so that, from (2.3.16),

$$j_r \notin \operatorname{supp}\left\{p_j^{[r]}\right\}, \quad r \geq r_0,$$

that is,

$$p_{j_r}^{[r]} = 0, \quad r \geq r_0,$$

from which it follows that, for $r \geq r_0$,

$$\left|\phi_{\boldsymbol{p}}\left(\tfrac{j_r}{3^r}\right)\right| = \left|\phi_{\boldsymbol{p}}\left(\tfrac{j_r}{3^r}\right) - p_{j_r}^{[r]}\right| \to 0, \quad r \to \infty,$$

from (2.4.10). It follows from the continuity of $\phi_{\boldsymbol{p}}$ at x, together with (2.4.9), that

$$\left|\phi_{\boldsymbol{p}}(x)\right| = \lim_{r\to\infty}\left|\phi_{\boldsymbol{p}}\left(\tfrac{j_r}{3^r}\right)\right| = 0,$$

so that

$$\phi_{\boldsymbol{p}}(x) = 0, \quad x \notin \left[\tfrac{\mu}{2}, \tfrac{\nu}{2}\right].$$

Moreover, by the continuity of $\phi_{\boldsymbol{p}}$ at $\frac{\mu}{2}$ and $\frac{\nu}{2}$, it follows that

$$\phi_{\boldsymbol{p}}\left(\tfrac{\mu}{2}\right) = 0; \quad \phi_{\boldsymbol{p}}\left(\tfrac{\nu}{2}\right) = 0,$$

so that

$$\phi_{\boldsymbol{p}}(x) = 0, \quad x \notin \left(\tfrac{\mu}{2}, \tfrac{\nu}{2}\right).$$

(ii) Let $x \in \mathbb{R}$. To prove the refinability of ϕ, we apply (2.3.15) to obtain, for $j \in \mathbb{Z}$ and $r = 2, 3, \ldots,$

$$\begin{aligned}
&\phi_{\boldsymbol{p}}\left(\tfrac{j_r}{3^r}\right) - \sum_k p_k \phi_{\boldsymbol{p}}\left(\tfrac{j_r}{3^{r-1}} - k\right)\\
&= \phi_{\boldsymbol{p}}\left(\tfrac{j_r}{3^r}\right) - p_{j_r}^{[r]} + \sum_k p_k p_{j_r - 3^{r-1}k}^{[r-1]} - \sum_k p_k \phi_{\boldsymbol{p}}\left(\tfrac{j_r}{3^{r-1}} - k\right)\\
&= \left[\phi_{\boldsymbol{p}}\left(\tfrac{j_r}{3^r}\right) - p_{j_r}^{[r]}\right] + \sum_k p_k\left[p_{j_r - 3^{r-1}k}^{[r-1]} - \phi_{\boldsymbol{p}}\left(\tfrac{j_r - 3^{r-1}k}{3^{r-1}}\right)\right],
\end{aligned}$$

from which it follows, by using also (2.4.10), that

$$\begin{aligned}
&\left|\phi_{\boldsymbol{p}}\left(\tfrac{j_r}{3^r}\right) - \sum_k p_k \phi_{\boldsymbol{p}}\left(\tfrac{j_r}{3^{r-1}} - k\right)\right|\\
&\leq \left|\phi_{\boldsymbol{p}}\left(\tfrac{j_r}{3^r}\right) - p_{j_r}^{[r]}\right| + \sum_k |p_k|\left|p_{j_r - 3^{r-1}k}^{[r-1]} - \phi_{\boldsymbol{p}}\left(\tfrac{j_r - 3^{r-1}k}{3^{r-1}}\right)\right|
\end{aligned}$$

$$\leq \sup_j \left|\phi_{\boldsymbol{p}}\left(\tfrac{j}{3^r}\right) - p_j^{[r]}\right| + \sup_j \left|\phi_{\boldsymbol{p}}\left(\tfrac{j}{3^{r-1}}\right) - p_j^{[r-1]}\right| \left(\sum_k |p_k|\right)$$

$$\to 0 + 0\left(\sum_k |p_k|\right) = 0, \quad r \to \infty. \tag{2.4.11}$$

It then follows from the continuity of the functions $\phi_{\boldsymbol{p}}$ and $\sum_k p_k \phi_{\boldsymbol{p}}(3 \cdot -k)$ at x, together with (2.4.9) and (2.4.11), that

$$\left|\phi_{\boldsymbol{p}}(x) - \sum_k p_k \phi_{\boldsymbol{p}}(3x - k)\right| = \lim_{r\to\infty} \left|\phi_{\boldsymbol{p}}\left(\tfrac{j_r}{3^r}\right) - \sum_k p_k \phi_{\boldsymbol{p}}\left(\tfrac{j_r}{3^{r-1}} - k\right)\right| = 0,$$

so that

$$\phi_{\boldsymbol{p}}(x) = \sum_k p_k \phi_{\boldsymbol{p}}(3x - k), \quad x \in \mathbb{R},$$

that is, $\phi_{\boldsymbol{p}}$ is a refinable function with refinement sequence $\{p_j\}$.

(iii) Let $x \in \mathbb{R}$. Since Theorem 2.3.1 implies that the sequence $\{p_j\}$ satisfies the sum-rule condition (2.2.8), we may apply (2.3.18) in Theorem 2.3.2 (iv) to obtain, for $r \in \mathbb{N}$,

$$\begin{aligned}\sum_k \phi_{\boldsymbol{p}}\left(\tfrac{j_r}{3^r} - k\right) - 1 &= \sum_k \left[\phi_{\boldsymbol{p}}\left(\tfrac{j_r}{3^r} - k\right) - p_{j_r - 3^r k}^{[r]}\right] \\ &= \sum_{k=\mu_r}^{\nu_r} \left[\phi_{\boldsymbol{p}}\left(\tfrac{j_r - 3^r k}{3^r}\right) - p_{j_r - 3^r k}^{[r]}\right],\end{aligned}$$

with, from (2.4.5) and (2.3.16),

$$\left.\begin{aligned}\mu_r &:= \min\left\{\lceil 3^{-r} j_r - \tfrac{\nu}{2}\rceil, \lceil 3^{-r}\left(j_r + \tfrac{\nu}{2}\right) - \tfrac{\nu}{2}\rceil\right\};\\ \nu_r &:= \max\left\{\lfloor 3^{-r} j_r - \tfrac{\mu}{2}\rfloor, \lfloor 3^{-r}\left(j_r + \tfrac{\mu}{2}\right) - \tfrac{\mu}{2}\rfloor\right\}.\end{aligned}\right\} \tag{2.4.12}$$

Hence, by using also (2.4.10), we have

$$\begin{aligned}\left|\sum_k \phi_{\boldsymbol{p}}\left(\tfrac{j_r}{3^r} - k\right) - 1\right| &\leq \sum_{k=\mu_r}^{\nu_r} \left|\phi_{\boldsymbol{p}}\left(\tfrac{j_r - 3^r k}{3^r}\right) - p_{j_r - 3^r k}^{[r]}\right| \\ &\leq \sup_j \left|\phi_{\boldsymbol{p}}\left(\tfrac{j}{3^r}\right) - p_j^{[r]}\right| \sum_{k=\mu_r}^{\nu_r} (1) \\ &= \sup_j \left|\phi_{\boldsymbol{p}}\left(\tfrac{j}{3^r}\right) - p_j^{[r]}\right| (\nu_r - \mu_r + 1) \\ &\to (0)\left(\tfrac{1}{2}(\nu - \mu)\right) = 0, \quad r \to \infty,\end{aligned} \tag{2.4.13}$$

from (2.4.12). It then follows from the continuity of the function $\sum_k \phi_{\boldsymbol{p}}(\cdot - k)$ at x, together with (2.4.9) and (2.4.13), that

$$\left|\sum_k \phi_{\boldsymbol{p}}(x-k) - 1\right| = \lim_{r\to\infty}\left|\sum_k \phi_{\boldsymbol{p}}\left(\tfrac{j_r}{3^r} - k\right) - 1\right| = 0, \qquad (2.4.14)$$

and thereby yielding the desired partition of unity property (2.4.7).

(iv) Let $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in l(\mathbb{Z})$ be such that $\Delta\boldsymbol{c} \in l^\infty$. By applying (2.3.17), (2.2.15), (2.4.5), (2.3.16), (2.4.12), (2.4.7) and (2.3.18), we obtain, for any $j \in \mathbb{Z}$ and $r \in \mathbb{N}$,

$$\begin{aligned}
&\sum_k \boldsymbol{c}_k \phi_{\boldsymbol{p}}\left(\tfrac{j}{3^r} - k\right) - \boldsymbol{c}_j^r \\
&= \sum_k \boldsymbol{c}_k \phi_{\boldsymbol{p}}\left(\tfrac{j-3^r k}{3^r}\right) - \sum_k p^{[r]}_{j-3^r k}\boldsymbol{c}_k \\
&= \sum_{k=\mu_r}^{\nu_r}\left[\phi_{\boldsymbol{p}}\left(\tfrac{j-3^r k}{3^r}\right) - p^{[r]}_{j-3^r k}\right]\boldsymbol{c}_k \\
&= \sum_{k=\mu_r}^{\nu_r}\left[\phi_{\boldsymbol{p}}\left(\tfrac{j}{3^r} - k\right) - p^{[r]}_{j-3^r k}\right](\boldsymbol{c}_k - \boldsymbol{c}_{\mu_r}) \\
&= \sum_{k=\mu_r}^{\nu_r}\left[\phi_{\boldsymbol{p}}\left(\tfrac{j}{3^r} - k\right) - p^{[r]}_{j-3^r k}\right]\left(\sum_{l=\mu_r+1}^{k}(\Delta\boldsymbol{c})_l\right).
\end{aligned}$$

Hence, by using also (2.4.10), we have

$$\begin{aligned}
&\left|\sum_k \boldsymbol{c}_k \phi_{\boldsymbol{p}}\left(\tfrac{j}{3^r} - k\right) - \boldsymbol{c}_j^r\right| \\
&\leq \sum_{k=\mu_r}^{\nu_r}\left|\phi_{\boldsymbol{p}}\left(\tfrac{j-3^r k}{3^r}\right) - p^{[r]}_{j-3^r k}\right|\left(\sum_{l=\mu_r+1}^{k}|(\Delta\boldsymbol{c})_l|\right) \\
&\leq \sup_j \left|\phi_{\boldsymbol{p}}\left(\tfrac{j}{3^r}\right) - p^{[r]}_j\right| ||\Delta\boldsymbol{c}||_\infty (\nu_r - \mu_r)^2 \\
&\to (0)\,||\Delta\boldsymbol{c}||_\infty \left(\tfrac{1}{4}(\nu-\mu)^2\right) = 0, \quad r\to\infty,
\end{aligned}$$

from (2.4.12). It follows, together with (2.2.1), that

$$\sup_j\left|\boldsymbol{F_c}\left(\tfrac{j}{3^r}\right) - \boldsymbol{c}_j^r\right| = \sup_j\left|\sum_k \boldsymbol{c}_k\phi_{\boldsymbol{p}}\left(\tfrac{j}{3^r} - k\right) - \boldsymbol{c}_j^r\right| \to 0, \quad r\to\infty,$$

and thereby completing our proof. ■

We proceed to prove that the limit function of a convergent subdivision scheme is not only a refinable function, but also a scaling function. We will need the notion of the integral moment of a function in C_0.

Definition 2.4.3 *For any non-negative integer j, we define the j^{th} integral moment of a function $f \in C_0$ by*

$$m_j = m_{f,j} := \int_{-\infty}^{\infty} x^j f(x) dx. \tag{2.4.15}$$

We shall rely on the following two lemmas, the proofs of which are straightforward adaptations of the proofs of the analogous results in binary subdivision (see Lemmas 4.3.1 and 4.3.2, pp 145 and 146, [1]).

Lemma 2.4.4 *Let ϕ be a refinable function with refinement sequence $\{p_j\}$, with $\operatorname{supp}\{p_j\} = [\mu,\nu]\big|_{\mathbb{Z}}$. Then the sequence $\{m_j : j = 0, 1, \ldots\}$ of integral moments of ϕ satisfies the identity*

$$m_j = \frac{1}{3^{j+1}} \sum_{l=0}^{j} \binom{j}{l} \left[\sum_{k=\mu}^{\nu} k^{j-l} p_k\right] m_l, \quad j = 0, 1, \ldots. \tag{2.4.16}$$

Proof.
From (2.4.15), (2.1.3), and $\operatorname{supp}\{p_j\} = [\mu,\nu]\big|_{\mathbb{Z}}$, we have, for $j = 0, 1, \ldots,$

$$\begin{aligned}
m_j &= \int_{-\infty}^{\infty} x^j \phi(x)\, dx \\
&= \int_{-\infty}^{\infty} x^j \left(\sum_{k=\mu}^{\nu} p_k \phi(3x-k)\right) dx \\
&= \sum_{k=\mu}^{\nu} p_k \left(\int_{-\infty}^{\infty} x^j \phi(3x-k)\, dx\right) \\
&= \frac{1}{3} \sum_{k=\mu}^{\nu} p_k \left(\int_{-\infty}^{\infty} \left(\frac{x+k}{3}\right)^j \phi(x)\, dx\right) \\
&= \left(\frac{1}{3^{j+1}}\right) \sum_{k=\mu}^{\nu} p_k \left(\int_{-\infty}^{\infty} (x+k)^j \phi(x)\, dx\right) \\
&= \left(\frac{1}{3^{j+1}}\right) \sum_{k=\mu}^{\nu} p_k \left(\int_{-\infty}^{\infty} \sum_{l=0}^{j} \binom{j}{l} x^l k^{j-l} \phi(x)\, dx\right) \\
&= \left(\frac{1}{3^{j+1}}\right) \sum_{l=0}^{j} \binom{j}{l} \left[\sum_{k=\mu}^{\nu} k^{j-l} p_k\right] \left(\int_{-\infty}^{\infty} x^l \phi(x)\, dx\right)
\end{aligned}$$

$$= \left(\frac{1}{3^{j+1}}\right) \sum_{l=0}^{j} \binom{j}{l} \left[\sum_{k=\mu}^{\nu} k^{j-l} p_k\right] m_l,$$

and thereby completing our proof. ■

Lemma 2.4.5 *Let* $f \in C_0$*, with* $\text{supp}^c f = [a, b]$ *. If*

$$\int_{-\infty}^{\infty} x^j f(x) dx = 0, \quad j = 0, 1, \ldots, \tag{2.4.17}$$

then f *is the zero function.*

Proof.
Suppose, to the contrary, that f is non-trivial.

Let $\varepsilon > 0$. By the Weierstrass polynomial approximation theorem (see [3], Theorem 3.3.4, p 61), there exists a polynomial g such that

$$\max_{a \le x \le b} |f(x) - g(x)| < \frac{\varepsilon}{\int_a^b |f(x)| dx}. \tag{2.4.18}$$

We observe that, for any polynomial g, we have, from (2.4.17) and $\text{supp}^c f = [a, b]$,

$$\int_a^b f(x) g(x) dx = 0. \tag{2.4.19}$$

By using (2.4.19), together with (2.4.18), we obtain

$$\begin{aligned} 0 \le \int_a^b (f(x))^2 \, dx &= \left| \int_a^b f(x) \left(f(x) - g(x)\right) dx \right| \\ &\le \int_a^b |f(x)||f(x) - g(x)| dx \\ &\le \max_{a \le x \le b} |f(x) - g(x)| \int_a^b |f(x)| dx \\ &< \frac{\varepsilon}{\int_a^b |f(x)| dx} \int_a^b |f(x)| dx = \varepsilon, \end{aligned}$$

for any $\varepsilon > 0$, from which it follows that

$$\int_a^b (f(x))^2 \, dx = 0.$$

By the continuity of f, it follows that $f = 0, \; x \in [a, b]$, which contradicts our assumption. Hence f must be the zero function. ■

By using the results from Lemma 2.4.4 and Lemma 2.4.5, we prove the following theorem, which in turn will be used to show that the limit function $\phi_{\boldsymbol{p}}$ of a convergent subdivision scheme is a scaling function, analogous to Theorem 4.3.3, p 149, [1]. The proof is similar to the proof of the analogous result for binary subdivision schemes (see Theorem 4.3.2, p 147, [1]), with, in particular, the proof of (iii) below representing a non-trivial adaptation of the binary case.

Theorem 2.4.6 *Let ϕ be a non-trivial refinable function with refinement sequence $\{p_j\}$. Then the following hold:*

(i) The condition

$$\int_{-\infty}^{\infty} \phi(x)dx \neq 0 \tag{2.4.20}$$

is satisfied if and only if

$$\sum_j p_j = 3; \tag{2.4.21}$$

(ii)

$$\sum_j p_j = 3^n \tag{2.4.22}$$

for some $n \in \mathbb{N}$;

(iii)

$$\int_{-\infty}^{\infty} \phi(x)dx = \sum_j \phi(j), \quad j \in \mathbb{Z}. \tag{2.4.23}$$

Proof.

(i) Suppose $\operatorname{supp}\{p_j\} = [\mu, \nu]\big|_{\mathbb{Z}}$. First, we prove that (2.4.20) implies (2.4.21). Suppose therefore that (2.4.20) holds. Note that, from (2.4.15) and (2.4.20), we have

$$m_0 = \int_{-\infty}^{\infty} \phi(x)dx \neq 0. \tag{2.4.24}$$

From (2.4.16) in Lemma 2.4.4, we have

$$m_0 = \tfrac{1}{3}\left(\sum_{k=\mu}^{\nu} p_k\right) m_0,$$

so that, by keeping in mind (2.4.24),

$$1 = \tfrac{1}{3}\left(\sum_{k=\mu}^{\nu} p_k\right),$$

that is,

$$\sum_{k=\mu}^{\nu} p_k = \sum_k p_k = 3.$$

To prove the converse, we assume that

$$\sum_j p_j = \sum_{j=\mu}^{\nu} p_j = 3, \tag{2.4.25}$$

and suppose, to the contrary, that

$$\int_{-\infty}^{\infty} \phi(x)dx = 0,$$

so that, by using also (2.4.15),

$$m_0 = \int_{-\infty}^{\infty} \phi(x)dx = 0. \tag{2.4.26}$$

Next, we observe that we can reformulate (2.4.16) in Lemma 2.4.4 as a recursive formulation, by writing m_j in terms of the lower order moments, yielding

$$\left(1 - \frac{1}{3^{j+1}} \sum_{k=\mu}^{\nu} p_k\right) m_j = \frac{1}{3^{j+1}} \sum_{l=0}^{j-1} \binom{j}{l} \left(\sum_{k=\mu}^{\nu} k^{j-l} p_k\right) m_l, \quad j = 1, 2, \ldots. \tag{2.4.27}$$

By using (2.4.25) in (2.4.27), we then obtain

$$m_j = \frac{1}{3\,(3^j - 1)} \sum_{l=0}^{j-1} \binom{j}{l} \left(\sum_{k=\mu}^{\nu} k^{j-l} p_k\right) m_l, \quad j = 1, 2, \ldots,$$

from which, together with (2.4.26), it follows that

$$m_j = 0, \quad j = 0, 1, \ldots.$$

It then follows from Lemma 2.4.5, together with (2.4.15), that ϕ must be the zero function, which contradicts our assumption that ϕ is non-trivial. Hence we must have

$$\int_{-\infty}^{\infty} \phi(x)dx \neq 0,$$

and thereby completing our proof.

(ii) Suppose, to the contrary, that there does not exist any integer $n \in \mathbb{N}$ such that (2.4.22) holds, according to which $\sum_j p_j \neq 3$, and it follows from (2.4.20), (2.4.21) in (i) that

$$\int_{-\infty}^{\infty} \phi(x)\, dx = 0.$$

Hence, by using also (2.4.15),

$$m_0 = \int_{-\infty}^{\infty} \phi(x)\, dx = 0. \tag{2.4.28}$$

Also, note from our assumption that (2.4.22) does not hold for any $n \in \mathbb{N}$, together with $\text{supp}\{p_j\} = [\mu, \nu]\big|_{\mathbb{Z}}$, that

$$1 - \frac{1}{3^{j+1}} \sum_{k=\mu}^{\nu} p_k \neq 0, \quad j = 1, 2, \ldots. \tag{2.4.29}$$

It follows from the recursive formulation (2.4.27), together with (2.4.28) and (2.4.29), that

$$m_j = 0, \quad j = 0, 1, \ldots,$$

so that Lemma 2.4.5 and (2.4.15) imply that ϕ must be the zero function, which contradicts our assumption that ϕ is non-trivial, and thus there exists some integer $n \in \mathbb{N}$ such that (2.4.22) holds.

(iii) Since $\text{supp}\{p_j\} = [\mu, \nu]\big|_{\mathbb{Z}}$, we know from (2.1.22) in Theorem 2.1.7 that

$$\text{supp}^c \phi = \left[\tfrac{1}{2}\mu, \tfrac{1}{2}\nu\right] \subset \left[\lfloor\left(\tfrac{\mu}{2}\right)\rfloor, \lceil\left(\tfrac{\nu}{2}\right)\rceil\right],$$

and thus

$$\phi(x) = 0, \quad x \in \mathbb{R} \setminus \left(\lfloor\left(\tfrac{\mu}{2}\right)\rfloor, \lceil\left(\tfrac{\nu}{2}\right)\rceil\right), \tag{2.4.30}$$

with also

$$\int_{-\infty}^{\infty} \phi(x) dx = \int_{\lfloor(\mu/2)\rfloor}^{\lceil(\nu/2)\rceil} \phi(x) dx. \tag{2.4.31}$$

Now observe that the sequence

$$x_j := \tfrac{j}{3^r}, \quad j = 3^r\lfloor\tfrac{\mu}{2}\rfloor, \ldots, 3^r\lceil\tfrac{\nu}{2}\rceil, \tag{2.4.32}$$

defines a uniform partition of the interval $\left[\lfloor\left(\tfrac{\mu}{2}\right)\rfloor, \lceil\left(\tfrac{\nu}{2}\right)\rceil\right]$, with

$$x_j - x_{j-1} = \tfrac{1}{3^r}, \quad j = 3^r\lfloor\tfrac{\mu}{2}\rfloor + 1, \ldots, 3^r\lceil\tfrac{\nu}{2}\rceil. \tag{2.4.33}$$

It therefore follows from the definition in terms of the limit of a Riemann sum of a definite integral that

$$
\begin{aligned}
\int_{\lfloor(\mu/2)\rfloor}^{\lceil(\nu/2)\rceil} \phi(x)dx &= \lim_{r\to\infty} \sum_{j=3^r\lfloor\mu/2\rfloor}^{3^r\lfloor\nu/2\rfloor-1} \left(\tfrac{1}{3^r}\right)\phi\left(x_j\right) \\
&= \lim_{r\to\infty} \sum_{j=3^r\lfloor\mu/2\rfloor}^{3^r\lfloor\nu/2\rfloor-1} \left(\tfrac{1}{3^r}\right)\phi\left(\tfrac{j}{3^r}\right) \\
&= \lim_{r\to\infty} \sum_{j} \left(\tfrac{1}{3^r}\right)\phi\left(\tfrac{j}{3^r}\right), \qquad (2.4.34)
\end{aligned}
$$

from (2.4.30). Hence, by combining (2.4.31) and (2.4.34), we deduce that

$$
\int_{-\infty}^{\infty} \phi(x)dx = \lim_{r\to\infty} \sum_{j} \left(\tfrac{1}{3^r}\right)\phi\left(\tfrac{j}{3^r}\right). \qquad (2.4.35)
$$

Now, for each $r \in \mathbb{N}$, we deduce, by repeated applications of (2.1.3), that

$$
\begin{aligned}
\sum_j \phi\left(\tfrac{j}{3^r}\right) &= \sum_j \left[\sum_{k_1} p_{k_1}\phi\left(\tfrac{j}{3^{r-1}} - k_1\right)\right] \\
&= \sum_j \sum_{k_1} p_{k_1} \left[\sum_{k_2} p_{k_2}\phi\left(\tfrac{j}{3^{r-2}} - 3k_1 - k_2\right)\right] \\
&= \sum_j \sum_{k_1} p_{k_1} \sum_{k_2} p_{k_2-3k_1}\phi\left(\tfrac{j}{3^{r-2}} - k_2\right) \\
&= \cdots \\
&= \sum_j \sum_{k_1} p_{k_1} \sum_{k_2} p_{k_2-3k_1} \sum_{k_3} p_{k_3-3k_2} \cdots \sum_{k_r} p_{k_r-3k_{r-1}}\phi\left(j - k_r\right) \\
&= \left(\sum_k p_k\right)^r \left(\sum_j \phi\left(j\right)\right). \qquad (2.4.36)
\end{aligned}
$$

By using (2.4.36), as well as (2.4.22) in (ii), in (2.4.35), we obtain

$$
\begin{aligned}
\int_{-\infty}^{\infty} \phi(x)dx &= \lim_{r\to\infty} \left(\tfrac{1}{3^r}\right) \left(\sum_k p_k\right)^r \left(\sum_j \phi\left(j\right)\right) \\
&= \left[\lim_{r\to\infty} \left(\tfrac{1}{3^r}\right)\left(3^n\right)^r\right] \sum_j \phi\left(j\right) \\
&= \left[\lim_{r\to\infty} \left(3^{n-1}\right)^r\right] \sum_j \phi\left(j\right), \qquad (2.4.37)
\end{aligned}
$$

for some $n \in \mathbb{N}$.

To prove (2.4.23), we observe, from (2.4.37), that if $n = 1$, we are done. If $n \geq 2$, it follows, by using also (2.4.22), that

$$\sum_j p_j = 3^n \neq 3,$$

so that, from (2.4.20), (2.4.21) in (i), we have

$$\int_{-\infty}^{\infty} \phi(x)dx = 0. \tag{2.4.38}$$

By substituting (2.4.38) into (2.4.37), we deduce that

$$\sum_j \phi(j) = 0. \tag{2.4.39}$$

Our result (2.4.23) follows by combining (2.4.38) and (2.4.39). ■

Theorem 2.4.7 *Let $\boldsymbol{p} = \{p_j\} \in l_0$ be such that the corresponding subdivision operator $\mathcal{S}_{\boldsymbol{p}}$ provides a convergent subdivision scheme with limit function $\phi_{\boldsymbol{p}}$. Then $\phi_{\boldsymbol{p}}$ is a scaling function, that is, $\phi_{\boldsymbol{p}}$ is a refinable function, with*

$$\int_{-\infty}^{\infty} \phi_{\boldsymbol{p}}(x)\, dx = 1. \tag{2.4.40}$$

Proof.

Since we have already shown in Theorem 2.4.2 (ii) that $\phi_{\boldsymbol{p}}$ is a refinable function, it only remains to prove that (2.4.40) holds. By Theorem 2.4.2 (iii), $\phi_{\boldsymbol{p}}$ provides a partition of unity, that is,

$$\sum_j \phi_{\boldsymbol{p}}(x-j) = 1, \quad x \in \mathbb{R},$$

so that we may set $x = 0$ to obtain

$$1 = \sum_j \phi_{\boldsymbol{p}}(0-j) = \sum_j \phi_{\boldsymbol{p}}(-j) = \sum_j \phi_{\boldsymbol{p}}(j)\,. \tag{2.4.41}$$

The result (2.4.40) then follows by combining (2.4.41) and (2.4.23) in Theorem 2.4.6 (iii). ■

We proceed to prove that the limit function $\phi_{\boldsymbol{p}}$ is the only function in C_0 that satisfies both (2.1.3) and (2.1.4), analogous to the result in Corollary 4.5.1, p 161, in [1]. We shall rely on the following uniqueness result, the proof of which is a straightforward adaptation of the proof of the analogous result in binary subdivision (see Theorem 4.5.1, p 160, [1]).

Theorem 2.4.8 *Let ϕ and $\tilde{\phi}$ be refinable functions such that*

$$\phi(x) = \sum_j p_j \phi(3x - j); \quad \tilde{\phi}(x) = \sum_j p_j \tilde{\phi}(3x - j), \tag{2.4.42}$$

that is, $\{p_j\}$ is the refinement sequence of both ϕ and $\tilde{\phi}$, with

$$\sum_j p_j = 3. \tag{2.4.43}$$

If, moreover,

$$\int_{-\infty}^{\infty} \phi(x)dx = \int_{-\infty}^{\infty} \tilde{\phi}(x)dx, \tag{2.4.44}$$

then

$$\phi = \tilde{\phi}. \tag{2.4.45}$$

Proof.
Suppose, to the contrary, that $\phi \neq \tilde{\phi}$, and define

$$\phi^* := \phi - \tilde{\phi} \neq 0. \tag{2.4.46}$$

By applying (2.4.46) and (2.4.42), we have

$$\begin{aligned}\sum_j p_j \phi^* (3x - j) &= \sum_j p_j \left[(\phi - \tilde{\phi})(3x - j) \right] \\ &= \sum_j p_j \phi(3x - j) - \sum_j p_j \tilde{\phi}(3x - j) \\ &= \phi(x) - \tilde{\phi}(x) = \phi^*(x),\end{aligned}$$

that is, ϕ^* is a (non-trivial) refinable function. Moreover, from (2.4.46) and (2.4.44), we have

$$\begin{aligned}\int_{-\infty}^{\infty} \phi^*(x)dx &= \int_{-\infty}^{\infty} (\phi - \tilde{\phi})(x)dx \\ &= \int_{-\infty}^{\infty} \phi(x)dx - \int_{-\infty}^{\infty} \tilde{\phi}(x)dx = 0,\end{aligned}$$

so that we may apply Theorem 2.4.6 (i) to deduce that

$$\sum_j p_j \neq 3,$$

which contradicts (2.4.43). We therefore conclude that $\phi = \tilde{\phi}$. ∎

Corollary 2.4.9 *Let $\{p_j\} \in l_0$ be such that*

$$\sum_j p_j = 3. \tag{2.4.47}$$

Then there exists at most one refinable function $\phi \in C_0$ with refinement sequence $\{p_j\}$, and such that ϕ provides a partition of unity.

Proof.

Suppose that ϕ and $\tilde{\phi}$ are both refinable functions with the same refinement sequence $\{p_j\}$, where $\sum\limits_j p_j = 3$, and that both ϕ and $\tilde{\phi}$ provide a partition of unity. Then we may apply (2.1.4) and (2.4.23) in Theorem 2.4.6 (iii) to obtain

$$1 = \sum_j \phi(0-j) = \sum_j \phi(-j) = \sum_j \phi(j) = \int_{-\infty}^{\infty} \phi(x)\, dx.$$

Similarly, we can also show that

$$\int_{-\infty}^{\infty} \tilde{\phi}(x)\, dx = 1,$$

so that we may apply Theorem 2.4.8 to conclude that $\phi = \tilde{\phi}$. ■

By combining Corollary 2.4.9 and Theorem 2.3.1, we have the following.

Corollary 2.4.10 *The limit function $\phi_{\mathbf{p}}$ of Theorem 2.4.2 is the only function in C_0 satisfying both*

$$\phi(x) = \sum_j p_j \phi(3x-j), \tag{2.4.48}$$

and

$$\sum_j \phi(x-j) = 1, \quad x \in \mathbb{R}. \tag{2.4.49}$$

We proceed to prove that if $\{p_j\}$ is a symmetric sequence, the corresponding refinable function ϕ will be a symmetric function, where we define the notion of symmetry in sequences and functions as follows.

Definition 2.4.11 *A sequence $\{c_j\} \in l_0$, with $\operatorname{supp}\{c_j\} = [j_0, j_1]\big|_{\mathbb{Z}}$, is said to be symmetric if*

$$c_{j_0+j} = c_{j_1-j}, \quad j \in \mathbb{Z}. \tag{2.4.50}$$

A function $g \in C_0$, with $\operatorname{supp}^c g = [\mu, \nu]$, is said to be symmetric if

$$g(\mu + x) = g(\nu - x), \quad x \in \mathbb{R}, \tag{2.4.51}$$

or equivalently,

$$g\left(\tfrac{1}{2}(\mu+\nu) - x\right) = g\left(\tfrac{1}{2}(\mu+\nu) + x\right), \quad x \in \mathbb{R}. \tag{2.4.52}$$

We shall rely on the following result with respect to integer shifts in a refinable function and its refinement sequence.

Theorem 2.4.12 *Let ϕ be a refinable function with refinement sequence $\{p_j\}$, and let ρ be any integer. Then the function*

$$\tilde{\phi}(x) := \phi(x+\rho), \quad x \in \mathbb{R}, \tag{2.4.53}$$

is refinable with refinement sequence $\{\tilde{p}_j\}$ given by

$$\tilde{p}_j := p_{j+2\rho}, \quad j \in \mathbb{Z}. \tag{2.4.54}$$

Proof. By using (2.1.3), (2.4.53) and (2.4.54), we obtain, for any $x \in \mathbb{R}$,

$$\begin{aligned}
\sum_j \tilde{p}_j \tilde{\phi}(3x - j) &= \sum_j p_{j+2\rho}\phi(3x - j + \rho) \\
&= \sum_j p_j \phi\left(3x - (j - 2\rho) + \rho\right) \\
&= \sum_j p_j \phi\left(3(x+\rho) - j\right) \\
&= \phi(x+\rho) = \tilde{\phi}(x).
\end{aligned}$$

∎

Note that, in contrast, the index transformation in the refinement sequence of a 2-refinable function agrees with the shift in the refinable function (see Lemma 4.5.1, p 161 in [1]).

We now apply the uniqueness result from Theorem 2.4.8 to show that symmetry in a refinement sequence $\{p_j\}$ is preserved by the corresponding refinable function ϕ. The proof follows the same pattern as the proof of the analogous result in binary subdivision (see Theorem 4.5.2, p 162, [1]), with some non-trivial extensions to obtain (2.4.60).

Theorem 2.4.13 *Let ϕ be a non-trivial refinable function with refinement sequence $\{p_j\}$, where $\{p_j\}$ is a symmetric sequence, with $\operatorname{supp}\{p_j\} = [\mu, \nu]\big|_{\mathbb{Z}}$, and such that*

$$\sum_j p_j = 3. \tag{2.4.55}$$

Then ϕ is a symmetric function, with

$$\phi\left(\tfrac{\mu}{2} + x\right) = \phi\left(\tfrac{\nu}{2} - x\right), \quad x \in \mathbb{R}. \tag{2.4.56}$$

Proof.
Define

$$\tilde{\phi} := \phi\left(\tfrac{\mu}{2} + x\right); \quad \tilde{\tilde{\phi}} := \phi\left(\tfrac{\nu}{2} - x\right), \tag{2.4.57}$$

and

$$\tilde{p_j} := p_{j+\mu}.$$

Since $\{p_j\}$ is a symmetric sequence, we have as in (2.4.50) that

$$\tilde{p_j} = p_{j+\mu} = p_{\nu-j}, \quad j \in \mathbb{Z}. \tag{2.4.58}$$

It follows from Theorem 2.4.12 that $\{\tilde{p_j}\}$ is the refinement sequence of $\tilde{\phi}$, that is,

$$\tilde{\phi}(x) = \sum_j \tilde{p_j}\tilde{\phi}\left(3x - j\right).$$

Now define

$$\phi^*(x) := \phi(x + \tfrac{\nu}{2}); \quad p_j^* := p_{j+\nu}, \tag{2.4.59}$$

so that it follows from Theorem 2.4.12 that ϕ^* is a refinable function with refinement sequence $\{p_j^*\}$, that is,

$$\phi^*(x) = \sum_j p_j^*\phi^*\left(3x - j\right).$$

This implies, by using also (2.4.59), that

$$\phi(x + \tfrac{\nu}{2}) = \sum_j p_{j+\nu}\phi\left(3x - j + \tfrac{\nu}{2}\right),$$

and thus, from (2.4.57) and (2.4.58),

$$\begin{aligned} \tilde{\tilde{\phi}}(x) = \phi(\tfrac{\nu}{2} - x) &= \sum_j p_{j+\nu}\phi\left(-3x - j + \tfrac{\nu}{2}\right) \\ &= \sum_j p_{\nu-j}\phi\left(-3x + j + \tfrac{\nu}{2}\right) \\ &= \sum_j \tilde{p_j}\tilde{\tilde{\phi}}\left(3x - j\right), \end{aligned} \tag{2.4.60}$$

that is, $\tilde{\tilde{\phi}}$ is a refinable function with refinement sequence $\{\tilde{p_j}\}$. By using (2.4.55), it follows that

$$\sum_j p_{j+\mu} = 3; \quad \sum_j p_{\nu-j} = 3,$$

that is, from (2.4.58),

$$\sum_j \tilde{p_j} = 3.$$

Moreover, it follows, by using (2.4.57), that

$$\int_{-\infty}^{\infty} \tilde{\phi}(x)\,dx = \int_{-\infty}^{\infty} \phi\left(\tfrac{\mu}{2}+x\right) dx = \int_{-\infty}^{\infty} \phi(x)\,dx;$$
$$\int_{-\infty}^{\infty} \tilde{\tilde{\phi}}(x)\,dx = \int_{-\infty}^{\infty} \phi\left(\tfrac{\nu}{2}-x\right) dx = \int_{-\infty}^{\infty} \phi(x)\,dx,$$

that is,

$$\int_{-\infty}^{\infty} \tilde{\phi}(x)\,dx = \int_{-\infty}^{\infty} \tilde{\tilde{\phi}}(x)\,dx.$$

Hence, by Theorem 2.4.8, it follows that

$$\tilde{\phi} = \tilde{\tilde{\phi}},$$

that is, from (2.4.57),

$$\phi\left(\tfrac{\mu}{2}+x\right) = \phi\left(\tfrac{\nu}{2}-x\right).$$

■

By combining Theorems 2.4.13 and 2.3.1, we obtain the following symmetry result on the limit function of a convergent subdivision scheme.

Corollary 2.4.14 *Let $\boldsymbol{p} = \{p_j\} \in l_0$ be a symmetric sequence, and suppose that the corresponding subdivision operator $\mathcal{S}_{\boldsymbol{p}}$ provides a convergent subdivision scheme with limit function $\phi_{\boldsymbol{p}}$. Then $\phi_{\boldsymbol{p}}$ is a symmetric function.*

We end this section with a summary of the properties of the limit function of a convergent subdivision scheme, as follows from Theorem 2.4.2 and Corollaries 2.4.10 and 2.4.14.

Corollary 2.4.15 *Let $\boldsymbol{p} = \{p_j\} \in l_0$, with $\operatorname{supp}\{p_j\} = [\mu,\nu]\big|_{\mathbb{Z}}$. Also, suppose that the corresponding subdivision operator $\mathcal{S}_{\boldsymbol{p}}$ provides a convergent subdivision scheme, and let $\phi_{\boldsymbol{p}}$ denote the limit function. Then we have the following:*

(i) $\phi_{\boldsymbol{p}}$ is finitely supported, with $\operatorname{supp}^c \phi_{\boldsymbol{p}} = \left[\frac{\mu}{2}, \frac{\nu}{2}\right]$;

(ii) $\phi_{\boldsymbol{p}}$ is a scaling function, that is, $\phi_{\boldsymbol{p}}$ is refinable with refinement sequence $\{p_j\}$, and $\phi_{\boldsymbol{p}}$ has unit integral;

(iii) $\phi_{\boldsymbol{p}}$ provides a partition of unity;

(iv) $\phi_{\boldsymbol{p}}$ is the only function in C_0 that is refinable with respect to the refinement sequence $\{p_j\}$, and that provides a partition of unity;

(v) If $\{p_j\}$ is a symmetric sequence, then $\phi_{\boldsymbol{p}}$ is a symmetric function.

Chapter 3

Construction of refinement sequence

In the rest of this thesis, our focus will be on the explicit construction of a sequence $\{p_j\} \in l_0$ such that its corresponding subdivision operator $\mathcal{S}_{\boldsymbol{p}}$ in (2.2.14) provides a convergent ternary interpolatory subdivision scheme, and for which interpolatory refinable functions form the basis functions of the limit curve (2.2.1), as discussed in Section 2.2.

First, in Section 3.1 below, for any refinable function ϕ with refinement sequence $\{p_j\}$, we shall derive, in the form of a Laurent polynomial identity, a necessary condition on $\{p_j\}$ for ϕ to be interpolatory, as in (2.2.12). To complete the argument, we will derive, in the next chapter, a subdivision convergence criterion which, when satisfied, will ensure the existence of an interpolatory refinable basis function.

We will proceed in Section 3.2 to explicitly construct a minimally supported sequence $\{p_j\}$ satisfying this necessary condition, before deriving, in Section 3.3, certain properties of $\{p_j\}$.

3.1 Necessary condition

We start by deriving a necessary condition on the refinement sequence of a refinable function ϕ, such that ϕ is interpolatory, as defined in Definition 2.2.2. The proof is a straightforward extension of the proof of the analogous result in binary subdivision (see Theorem 8.1.3, p 298 in [1]).

Theorem 3.1.1 *Let ϕ be a refinable function with refinement sequence $\{p_j\}$, and suppose that ϕ is a canonical interpolant on $\mathbb{Z}$, that is,*

$$\phi(j) = \delta_j, \quad j \in \mathbb{Z}, \tag{3.1.1}$$

with $\{\delta_j\}$ *denoting the Kronecker delta sequence, as in (2.2.13). Then* $\{p_j\}$ *satisfies the property*

$$p_{3j} = \delta_j, \quad j \in \mathbb{Z}. \tag{3.1.2}$$

Proof.
By applying (3.1.1), (2.1.3) and (2.2.13), we obtain, for $j \in \mathbb{Z}$,

$$\delta_j = \phi(j) = \sum_k p_k \phi(3j - k) = \sum_k p_{3j-k}\phi(k) = \sum_k p_{3j-k}\delta_k = p_{3j},$$

so that (3.1.2) holds. ■

We can show that the subdivision operator corresponding to a refinement sequence satisfying (3.1.2), is indeed an interpolatory subdivision operator, as defined in Definition 2.2.4, as follows. The proof is a straightforward adaptation of the proof of the analogous result in binary subdivision (see Theorem 8.1.1, p 296, [1]).

Theorem 3.1.2 *A sequence* $\{p_j\} \in l_0$ *satisfies (3.1.2) if and only if the corresponding subdivision operator* $\mathcal{S}_{\boldsymbol{p}}$*, defined by (2.2.14), is an interpolatory subdivision operator.*

Proof.
We first assume $\{p_j\} \in l_0$ is such that $\mathcal{S}_{\boldsymbol{p}}$ is an interpolatory subdivision operator, that is,

$$(\mathcal{S}_{\boldsymbol{p}}\boldsymbol{c})_{3j} = \boldsymbol{c}_j, \quad j \in \mathbb{Z}, \tag{3.1.3}$$

for any $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in l(\mathbb{Z})$. It follows from (3.1.3) and (2.2.14), together with (2.2.13), that

$$\delta_j = (\mathcal{S}_{\boldsymbol{p}}\boldsymbol{\delta})_{3j} = \sum_k p_{3j-3k}\delta_k = p_{3j},$$

for all $j \in \mathbb{Z}$. To prove the converse, we assume $\{p_j\} \in l_0$ is such that (3.1.2) is satisfied. Let $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in l(\mathbb{Z})$. By using (2.2.14), (3.1.2) and (2.2.13), we obtain

$$(\mathcal{S}_{\boldsymbol{p}}\boldsymbol{c})_{3j} = \sum_k p_{3j-3k}\boldsymbol{c}_k = \sum_k p_{3k}\boldsymbol{c}_{j-k} = \sum_k \delta_k \boldsymbol{c}_{j-k} = \boldsymbol{c}_j,$$

for all $j \in \mathbb{Z}$, and thereby completing our proof. ■

Definition 3.1.3 *We define the (three-scale) symbol of a sequence* $\{p_j\} \in l_0$ *to be the Laurent polynomial*

$$P(z) := \frac{1}{3}\sum_j p_j z^j, \quad z \in \mathbb{C} \setminus \{0\}. \tag{3.1.4}$$

Furthermore, if $\{p_j\}$ *is the refinement sequence of some refinable function* ϕ*, then we refer to* P *as the (three-scale) symbol of* ϕ*.*

The necessary condition (3.1.2) on the refinement sequence of an interpolatory refinable function, has the following equivalent formulation, in terms of the Laurent polynomial symbol of $\{p_j\}$, and in which we introduce the complex number

$$\alpha := e^{2\pi i/3}, \tag{3.1.5}$$

for which it holds that

$$\alpha^{3j} = 1, \quad j \in \mathbb{Z}; \tag{3.1.6}$$
$$\alpha^2 + \alpha + 1 = 0. \tag{3.1.7}$$

The result here is a non-trivial extension of the analogous result in binary subdivision (Theorem 8.1.1, p 296 in [1]).

Theorem 3.1.4 *A sequence $\{p_j\} \in l_0$ satisfies (3.1.2) if and only if its three-scale symbol P, as defined by (3.1.4), satisfies the Laurent polynomial identity*

$$P(z) + P(\alpha z) + P(\alpha^2 z) = 1, \quad z \in \mathbb{C} \setminus \{0\}, \tag{3.1.8}$$

where α is the complex number defined by (3.1.5).

Proof.
We start by applying (3.1.4), (3.1.6) and (3.1.7) to obtain, for $z \in \mathbb{C} \setminus \{0\}$,

$$\begin{aligned}
&P(z) + P(\alpha z) + P(\alpha^2 z) \\
&= \tfrac{1}{3} \sum_j p_j z^j + \tfrac{1}{3} \sum_j p_j \alpha^j z^j + \tfrac{1}{3} \sum_j p_j \alpha^{2j} z^j \\
&= \tfrac{1}{3} \left[\sum_j p_{3j} z^{3j} + \sum_j p_{3j+1} z^{3j+1} + \sum_j p_{3j+2} z^{3j+2} \right] \\
&\quad + \tfrac{1}{3} \left[\sum_j p_{3j} \alpha^{3j} z^{3j} + \sum_j p_{3j+1} \alpha^{3j+1} z^{3j+1} + \sum_j p_{3j+2} \alpha^{3j+2} z^{3j+2} \right] \\
&\quad + \tfrac{1}{3} \left[\sum_j p_{3j} \alpha^{6j} z^{3j} + \sum_j p_{3j+1} \alpha^{6j+2} z^{3j+1} + \sum_j p_{3j+2} \alpha^{6j+4} z^{3j+2} \right] \\
&= \tfrac{1}{3} \left[\sum_j p_{3j} \left(1+1+1\right) z^{3j} \right] + \tfrac{1}{3} \left[\sum_j p_{3j+1} \left(1 + \alpha + \alpha^2\right) z^{3j+1} \right] \\
&\quad + \tfrac{1}{3} \left[\sum_j p_{3j+2} \left(1 + \alpha^2 + \alpha\right) z^{3j+2} \right] = \sum_j p_{3j} z^{3j}.
\end{aligned} \tag{3.1.9}$$

It then immediately follows from (3.1.9) that the two conditions (3.1.2) and (3.1.8) are equivalent. ■

It follows from Theorems 3.1.1 and 3.1.4 that a necessary condition on the refinement sequence $\{p_j\}$ such that its refinable function ϕ satisfies the interpolatory condition (3.1.1), is that its symbol P must satisfy the polynomial identity (3.1.8).

It is natural to require that our interpolatory subdivision scheme satisfies, for some integer $m \in \mathbb{N}$, the polynomial reproduction property

$$\sum_j f(j)\phi_{\boldsymbol{p}}(x-j) = f(x), \quad x \in \mathbb{R}, \quad f \in \pi_{m-1}, \tag{3.1.10}$$

where $\phi_{\boldsymbol{p}}$ denotes the corresponding limit function, as described in Corollary 2.4.15, thereby ensuring that if the initial control points are chosen to lie on some parametric polynomial $f \in \pi_{m-1}$, the limit curve (2.2.1) is precisely f. As we will show in the remainder of this section, a sufficient condition on the corresponding refinement sequence $\{p_j\}$ for the polynomial reproduction property (3.1.10) to be achieved, is that $\{p_j\}$ satisfies the sum-rule condition of order at least m. The sum-rule condition of order m is defined as follows.

Definition 3.1.5 *Let $m \in \mathbb{N}$. A sequence $\{p_j\} \in l_0$ is said to satisfy the m^{th} order sum-rule condition if m is the largest integer such that*

$$\left.\begin{array}{l} \beta_l := \sum_j (3j)^l p_{3j} = \sum_j (3j-1)^l p_{3j-1} = \sum_j (3j-2)^l p_{3j-2}, \quad l = 0,\ldots,m-1; \\ \textit{with } \beta_0 = 1, \end{array}\right\} \tag{3.1.11}$$

and where $0^0 := 1$.

We also define the space of discrete polynomials as follows.

Definition 3.1.6 *For a non-negative integer k, the space of discrete polynomials of degree $\leq k$ is defined by*

$$\pi_k^d := \{\{c_j\} \in l(\mathbb{Z}) : c_j = f(j),\ j \in \mathbb{Z},\ f \in \pi_k\}. \tag{3.1.12}$$

Our first result is to show that if $\{p_j\}$ satisfies the sum-rule condition of order at least m, the subdivision operator $\mathcal{S}_{\boldsymbol{p}}$ maps the discrete polynomial space π_{m-1}^d into itself, as follows. The proof is similar to the proof of the analogous result for binary subdivision (see Theorem 5.1.1, p 170 in [1]).

We shall use the notation

$$e_j^k := j^k, \quad j \in \mathbb{Z}, \tag{3.1.13}$$

to denote the discrete monomial $\boldsymbol{e}^k := \left\{e_j^k\right\}$ in π_k^d.

Lemma 3.1.7 *For $m \in \mathbb{N}$, let $\boldsymbol{p} = \{p_j\} \in l_0$ be such that $\{p_j\}$ satisfies the sum-rule condition of order at least m. Then, for $l = 0, \ldots, m-1$,*

$$\left(\mathcal{S}_{\boldsymbol{p}} \boldsymbol{e}^l\right)_j = f_l(j), \quad j \in \mathbb{Z}, \tag{3.1.14}$$

where $f_l \in \pi_l \subset \pi_{m-1}$ is defined by

$$f_l(x) := \sum_{j=0}^{l} f_j^l x^j, \tag{3.1.15}$$

with

$$f_j^l := \tfrac{1}{3^l}(-1)^{l-j}\tbinom{l}{j}\beta_{l-j}, \quad j \in \mathbb{Z}, \tag{3.1.16}$$

and where $\{\beta_l : l = 0, \ldots, m-1\}$ is defined as in (3.1.11).

Proof.

Let $l \in \{0, \ldots, m-1\}$. By applying (2.2.14), (3.1.13) and (3.1.11), we obtain, for $j \in \mathbb{Z}$,

$$\begin{aligned}
\left(\mathcal{S}_{\boldsymbol{p}} \boldsymbol{e}^l\right)_{3j} &= \sum_k p_{3j-3k} k^l \\
&= \sum_k p_{3k} \left(j-k\right)^l \\
&= \tfrac{1}{3^l} \sum_k p_{3k} \left(3j-3k\right)^l \\
&= \tfrac{1}{3^l} \sum_k p_{3k} \sum_{n=0}^{l} \tbinom{l}{n} (3j)^n (-1)^{l-n} (3k)^{l-n} \\
&= \tfrac{1}{3^l} \sum_{n=0}^{l} \tbinom{l}{n} (3j)^n (-1)^{l-n} \left[\sum_k p_{3k} (3k)^{l-n} \right] \\
&= \tfrac{1}{3^l} \sum_{n=0}^{l} \tbinom{l}{n} (3j)^n (-1)^{l-n} \beta_{l-n},
\end{aligned} \tag{3.1.17}$$

whereas

$$\begin{aligned}
\left(\mathcal{S}_{\boldsymbol{p}} \boldsymbol{e}^l\right)_{3j-1} &= \sum_k p_{3j-1-3k} k^l \\
&= \sum_k p_{3k-1} \left(j-k\right)^l \\
&= \tfrac{1}{3^l} \sum_k p_{3k-1} \left(3j-3k\right)^l \\
&= \tfrac{1}{3^l} \sum_k p_{3k-1} \left[(3j-1)-(3k-1)\right]^l
\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{3^l} \sum_k p_{3k-1} \sum_{n=0}^{l} \tbinom{l}{n} (3j-1)^n (-1)^{l-n} (3k-1)^{l-n} \\
&= \tfrac{1}{3^l} \sum_{n=0}^{l} \tbinom{l}{n} (3j-1)^n (-1)^{l-n} \left[\sum_k p_{3k-1} (3k-1)^{l-n} \right] \\
&= \tfrac{1}{3^l} \sum_{n=0}^{l} \tbinom{l}{n} (3j-1)^n (-1)^{l-n} \beta_{l-n}, \qquad (3.1.18)
\end{aligned}
$$

and, similarly,

$$
\left(\mathcal{S}_{\boldsymbol{p}} \boldsymbol{e}^l\right)_{3j-2} = \tfrac{1}{3^l} \sum_{n=0}^{l} \tbinom{l}{n} (3j-2)^n (-1)^{l-n} \beta_{l-n}. \qquad (3.1.19)
$$

Combining (3.1.17), (3.1.18) and (3.1.19), we obtain

$$
\begin{aligned}
\left(\mathcal{S}_{\boldsymbol{p}} \boldsymbol{e}^l\right)_j &= \tfrac{1}{3^l} \sum_{n=0}^{l} \tbinom{l}{n} j^n (-1)^{l-n} \beta_{l-n} \\
&= \sum_{n=0}^{l} \left[\tfrac{1}{3^l} (-1)^{l-n} \tbinom{l}{n} \beta_{l-n} \right] j^n = \sum_{n=0}^{l} f_j^l j^n = f_l(j),
\end{aligned}
$$

from (3.1.15) and (3.1.16), so that (3.1.14) holds. ■

We proceed to introduce the concept of a discrete moment of a compactly supported function.

Definition 3.1.8 *Let $l \in \mathbb{N}$ and $f \in C_0$. We define the l^{th} order discrete moment of f by*

$$
\mu_l = \mu_{f,l} := \sum_j j^l f(j). \qquad (3.1.20)
$$

The following identity is satisfied by discrete moments of a refinable function. The proof is similar to the proof of the analogous result in binary subdivision (see Theorem 5.1.2, p 172 in [1]).

Lemma 3.1.9 *Let ϕ be a refinable function with corresponding refinement sequence $\{p_j\}$ satisfying the sum-rule condition of order at least $m \in \mathbb{N}$. Then the sequence of discrete moments $\{\mu_l : l = 0, \ldots, m-1\}$ of ϕ satisfies the identity*

$$
\mu_l = \tfrac{1}{3^l} \sum_{j=0}^{l} \tbinom{l}{j} \beta_{l-j} \mu_j, \quad l = 0, \ldots, m-1. \qquad (3.1.21)
$$

Proof.

By applying the definition (3.1.20), the refinement equation (2.1.3), as well as (3.1.11), we obtain, for $l = 0, \dots, m-1$,

$$\begin{aligned}
\mu_l &= \sum_j j^l \phi(j) \\
&= \sum_j j^l \left(\sum_k p_k \phi(3j-k) \right) \\
&= \sum_j j^l \left(\sum_k p_{3j-k} \phi(k) \right) \\
&= \sum_j j^l \left(\sum_k p_{3j-3k} \phi(3k) \right) + \sum_j j^l \left(\sum_k p_{3j-3k-1} \phi(3k+1) \right) \\
&\quad + \sum_j j^l \left(\sum_k p_{3j-3k-2} \phi(3k+2) \right) \\
&= \sum_k \sum_j (j+k)^l p_{3j} \phi(3k) + \sum_k \sum_j (j+k)^l p_{3j-1} \phi(3k+1) \\
&\quad + \sum_k \sum_j (j+k)^l p_{3j-2} \phi(3k+2) \\
&= \tfrac{1}{3^l} \sum_k \sum_j (3j+3k)^l p_{3j} \phi(3k) + \tfrac{1}{3^l} \sum_k \sum_j \left((3j-1)+(3k+1)\right)^l p_{3j-1} \phi(3k+1) \\
&\quad + \tfrac{1}{3^l} \sum_k \sum_j \left((3j-2)+(3k+2)\right)^l p_{3j-2} \phi(3k+2) \\
&= \tfrac{1}{3^l} \sum_k \sum_j \sum_{n=0}^{l} \tbinom{l}{n} (3j)^{l-n} (3k)^n p_{3j} \phi(3k) \\
&\quad + \tfrac{1}{3^l} \sum_k \sum_j \sum_{n=0}^{l} \tbinom{l}{n} (3j-1)^{l-n} (3k+1)^n p_{3j-1} \phi(3k+1) \\
&\quad + \tfrac{1}{3^l} \sum_k \sum_j \sum_{n=0}^{l} \tbinom{l}{n} (3j-2)^{l-n} (3k+2)^n p_{3j-2} \phi(3k+2) \\
&= \tfrac{1}{3^l} \sum_{n=0}^{l} \tbinom{l}{n} \sum_k (3k)^n \phi(3k) \beta_{l-n} + \tfrac{1}{3^l} \sum_{n=0}^{l} \tbinom{l}{n} \sum_k (3k+1)^n \phi(3k+1) \beta_{l-n} \\
&\quad + \tfrac{1}{3^l} \sum_{n=0}^{l} \tbinom{l}{n} \sum_k (3k+2)^n \phi(3k+2) \beta_{l-n} \\
&= \tfrac{1}{3^l} \sum_{n=0}^{l} \tbinom{l}{n} \beta_{l-n} \left[\sum_k k^n \phi(k) \right] = \tfrac{1}{3^l} \sum_{n=0}^{l} \tbinom{l}{n} \beta_{l-n} \mu_n,
\end{aligned}$$

so that (3.1.21) holds. ■

By using Lemmas 3.1.7 and 3.1.9, we can derive the following "commutator identity", which will be necessary to prove the polynomial reproduction property (3.1.10) of our interpolatory subdivision scheme. The proof follows the same pattern as the proof of the analogous result for binary subdivision (see Theorem 5.2.1, p 173, [1]).

Theorem 3.1.10 *Let ϕ be a refinable function with corresponding refinement sequence $\boldsymbol{p} = \{p_j\}$ satisfying the sum-rule condition of order at least $m \in \mathbb{N}$. Then*

$$\sum_j f(j)\phi(x-j) = \sum_j \phi(j) f(x-j), \quad x \in \mathbb{R}, \quad f \in \pi_{m-1}. \tag{3.1.22}$$

Proof.
Since the sequence $\left\{\frac{k}{3^r} : k \in \mathbb{Z},\ r = 0, 1, \ldots\right\}$ is dense in $\mathbb{R}$ and ϕ is a continuous function on $\mathbb{R}$, it suffices to show, for $k \in \mathbb{Z}$ and $r = 0, 1, \ldots$, that

$$\sum_j f(j)\phi(\tfrac{k}{3^r} - j) = \sum_j \phi(j) f(\tfrac{k}{3^r} - j), \tag{3.1.23}$$

with $f \in \pi_{m-1}$, or, equivalently,

$$\sum_j j^l \phi(\tfrac{k}{3^r} - j) = \sum_j \phi(j)(\tfrac{k}{3^r} - j)^l, \quad l = 0, \ldots, m-1. \tag{3.1.24}$$

We proceed to prove (3.1.24) by induction on r. If $r = 0$, (3.1.24) holds trivially. We now assume the result holds for some non-negative integer r. By applying consecutively (2.1.3), (2.2.14), (3.1.13) and (3.1.14) in Lemma 3.1.7, the induction hypothesis as in (3.1.23), (3.1.15), (3.1.20) and (3.1.16), we obtain, for $l = 0, \ldots, m-1$,

$$\begin{aligned}
\sum_j j^l \phi\left(\tfrac{k}{3^{r+1}} - j\right) &= \sum_j j^l \left(\sum_n p_n \phi\left(\tfrac{k}{3^r} - 3j - n\right)\right) \\
&= \sum_j j^l \sum_n p_{n-3j} \phi\left(\tfrac{k}{3^r} - n\right) \\
&= \sum_n \left(\sum_j p_{n-3j} j^l\right) \phi\left(\tfrac{k}{3^r} - n\right) \\
&= \sum_n \left(\mathcal{S}_{\boldsymbol{p}} \boldsymbol{e}^l\right)_n \phi\left(\tfrac{k}{3^r} - n\right) \\
&= \sum_n f_l(n) \phi\left(\tfrac{k}{3^r} - n\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_n \phi(n) f_l \left(\tfrac{k}{3^r} - n\right) \\
&= \sum_n \phi(n) \sum_j f_j^l \left(\tfrac{k}{3^r} - n\right)^j \\
&= \sum_n \phi(n) \sum_j f_j^l \sum_i \binom{j}{i} \left(\tfrac{k}{3^r}\right)^i (-1)^{j-i} n^{j-i} \\
&= \sum_j f_j^l \sum_i \binom{j}{i} \left(\tfrac{k}{3^r}\right)^i (-1)^{j-i} \mu_{j-i} \\
&= \sum_i (-1)^i \left(\tfrac{k}{3^r}\right)^i \sum_j (-1)^j \binom{j}{i} f_j^l \mu_{j-i} \\
&= \sum_i (-1)^i \left(\tfrac{k}{3^r}\right)^i \sum_j (-1)^j \binom{j}{i} \left[\tfrac{1}{3^l}(-1)^{l-j}\binom{l}{j}\beta_{l-j}\right] \mu_{j-i} \\
&= \tfrac{(-1)^l}{3^l} \sum_i (-1)^i \left(\tfrac{k}{3^r}\right)^i \sum_j \binom{j}{i}\binom{l}{j}\beta_{l-j}\mu_{j-i}. \qquad (3.1.25)
\end{aligned}
$$

Next, we apply (3.1.20) and (3.1.21) in Lemma 3.1.9 to obtain, for $l = 0, \ldots, m-1$,

$$
\begin{aligned}
\sum_j \phi(j) \left(\tfrac{k}{3^{r+1}} - j\right)^l &= \sum_j \phi(j) \sum_n \binom{l}{n} \left(\tfrac{k}{3^{r+1}}\right)^n (-1)^{l-n} j^{l-n} \\
&= (-1)^l \sum_n (-1)^n \tfrac{1}{3^n} \left(\tfrac{k}{3^r}\right)^n \binom{l}{n} \mu_{l-n} \\
&= (-1)^l \sum_n (-1)^n \tfrac{1}{3^n} \left(\tfrac{k}{3^r}\right)^n \binom{l}{n} \left[\tfrac{1}{3^{l-n}} \sum_i \binom{l-n}{i} \beta_{l-n-i}\mu_i\right] \\
&= \tfrac{(-1)^l}{3^l} \sum_n (-1)^n \left(\tfrac{k}{3^r}\right)^n \binom{l}{n} \sum_i \binom{l-n}{i-n} \beta_{l-i}\mu_{i-n} \\
&= \tfrac{(-1)^l}{3^l} \sum_i (-1)^i \left(\tfrac{k}{3^r}\right)^i \binom{l}{i} \sum_j \binom{l-i}{j-i} \beta_{l-j}\mu_{j-i}. \qquad (3.1.26)
\end{aligned}
$$

The proof is completed by observing that

$$\binom{j}{i}\binom{l}{j} = \binom{l}{i}\binom{l-i}{j-i}, \qquad (3.1.27)$$

as follows from

$$
\begin{aligned}
\binom{j}{i}\binom{l}{j} &= \tfrac{j!}{i!(j-i)!}\tfrac{l!}{j!(l-j)!} = \tfrac{l!}{i!(j-i)!(l-j)!}; \qquad (3.1.28) \\
\binom{l}{i}\binom{l-i}{j-i} &= \tfrac{l!}{i!(l-i)!}\tfrac{(l-i)!}{(j-i)!(l-j)!} = \tfrac{l!}{i!(j-i)!(l-j)!}, \qquad (3.1.29)
\end{aligned}
$$

so that (3.1.24) follows by combining (3.1.25) and (3.1.26). ∎

By applying the identity (3.1.22) in Theorem 3.1.10, we now immediately deduce the following polynomial reproduction result.

Corollary 3.1.11 *Let ϕ denote a refinable function with refinement sequence $\{p_j\}$ satisfying the sum-rule condition of order at least $m \in \mathbb{N}$. Suppose, moreover, that ϕ satisfies the condition (3.1.1). Then*

$$\sum_j f(j)\phi_{\boldsymbol{p}}(x-j) = f(x), \quad x \in \mathbb{R}, \quad f \in \pi_{m-1}. \tag{3.1.30}$$

Recall from Theorem 3.1.1 that the condition (3.1.1) for ϕ implies the condition (3.1.2) for $\{p_j\}$.

The sum-rule condition of order at least m on the refinement sequence $\{p_j\}$ has the following equivalent formulation in terms of the Laurent polynomial symbol P of $\{p_j\}$. The proof is a non-trivial extension of the proof of the analogous result in binary subdivision (see Theorem 5.3.1, p 179 in [1]).

Theorem 3.1.12 *A sequence $\{p_j\} \in l_0$ satisfies the sum-rule condition of order at least $m \in \mathbb{N}$ if and only if its three-scale symbol P, as defined in (3.1.4), satisfies the formulation*

$$P(z) = \left(\frac{1+z+z^2}{3}\right)^m R(z), \quad z \in \mathbb{C} \setminus \{0\}, \tag{3.1.31}$$

where R is a Laurent polynomial such that

$$R(1) = 1, \tag{3.1.32}$$

with the complex number α defined by (3.1.5).

Proof.
We start by differentiating (3.1.4) repeatedly to obtain, for $l = 1, 2, \ldots$, and by using also (3.1.6),

$$\begin{aligned} P^{(l)}(\alpha) &= \tfrac{1}{3}\left[\sum_j w_l(3j)p_{3j}\alpha^{3j-l} + \sum_j w_l(3j-1)p_{3j-1}\alpha^{3j-l-1}\right. \\ &\quad \left. + \sum_j w_l(3j-2)p_{3j-2}\alpha^{3j-l-2}\right] \\ &= \tfrac{1}{3}\alpha^{2l}\left[\sum_j w_l(3j)p_{3j} + \alpha^2\sum_j w_l(3j-1)p_{3j-1} + \alpha\sum_j w_l(3j-2)p_{3j-2}\right], \end{aligned} \tag{3.1.33}$$

where

$$w_l(x) := \prod_{k=0}^{l-1}(x-k), \quad l = 1, 2, \ldots, \tag{3.1.34}$$

and, similarly,

$$P^{(l)}(\alpha^2) = \tfrac{1}{3}\alpha^l \left[\sum_j w_l(3j)p_{3j} + \alpha \sum_j w_l(3j-1)p_{3j-1} + \alpha^2 \sum_j w_l(3j-2)p_{3j-2} \right]. \tag{3.1.35}$$

From (3.1.34), there exists a coefficient sequence $\{c_{l,k} : k = 0, \ldots, l-1\}$ such that

$$w_l(x) = x^l + \sum_{k=0}^{l-1} c_{l,k}x^k, \quad x \in \mathbb{R}, \tag{3.1.36}$$

from which it follows, together with (3.1.33) and (3.1.35), that

$$\begin{aligned} P^{(l)}(\alpha) = \tfrac{1}{3}\alpha^{2l} \Bigg[& \left(\sum_j (3j)^l p_{3j} + \alpha^2 \sum_j (3j-1)^l p_{3j-1} + \alpha \sum_j (3j-2)^l p_{3j-2} \right) \\ & + \sum_{k=0}^{l-1} c_{l,k} \left(\sum_j (3j)^k p_{3j} + \alpha^2 \sum_j (3j-1)^k p_{3j-1} + \alpha \sum_j (3j-2)^k p_{3j-2} \right) \Bigg], \end{aligned} \tag{3.1.37}$$

and

$$\begin{aligned} P^{(l)}(\alpha^2) = \tfrac{1}{3}\alpha^{l} \Bigg[& \left(\sum_j (3j)^l p_{3j} + \alpha \sum_j (3j-1)^l p_{3j-1} + \alpha^2 \sum_j (3j-2)^l p_{3j-2} \right) \\ & + \sum_{k=0}^{l-1} c_{l,k} \left(\sum_j (3j)^k p_{3j} + \alpha \sum_j (3j-1)^k p_{3j-1} + \alpha^2 \sum_j (3j-2)^k p_{3j-2} \right) \Bigg]. \end{aligned} \tag{3.1.38}$$

Next, we show that the sum-rule condition (2.2.8) has the equivalent formulation

$$P(1) = 1; \quad P(\alpha) = 0; \quad P(\alpha^2) = 0, \tag{3.1.39}$$

in terms of the symbol P of the refinement sequence $\{p_j\}$. To this end, we first note, from (3.1.4) and (3.1.6), that

$$\left.\begin{aligned} P(1) &= \tfrac{1}{3}\sum_j p_j(1)^j = \tfrac{1}{3}\sum_j p_{3j} + \tfrac{1}{3}\sum_j p_{3j-1} + \tfrac{1}{3}\sum_j p_{3j-2}; \\ P(\alpha) &= \tfrac{1}{3}\sum_j p_j(\alpha)^j = \tfrac{1}{3}\sum_j p_{3j} + \tfrac{1}{3}\sum_j p_{3j-1}\alpha^2 + \tfrac{1}{3}\sum_j p_{3j-2}\alpha; \\ P(\alpha^2) &= \tfrac{1}{3}\sum_j p_j(\alpha^2)^j = \tfrac{1}{3}\sum_j p_{3j} + \tfrac{1}{3}\sum_j p_{3j-1}\alpha + \tfrac{1}{3}\sum_j p_{3j-2}\alpha^2, \end{aligned}\right\} \tag{3.1.40}$$

that is,

$$\frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha^2 & \alpha \\ 1 & \alpha & \alpha^2 \end{bmatrix} \begin{bmatrix} \sum_j p_{3j} \\ \sum_j p_{3j-1} \\ \sum_j p_{3j-2} \end{bmatrix} = \begin{bmatrix} P(1) \\ P(\alpha) \\ P(\alpha^2) \end{bmatrix}. \tag{3.1.41}$$

Suppose $\{p_j\} \in l_0$ satisfies the sum-rule condition (2.2.8). It then follows from (3.1.40) and (3.1.7) that (3.1.39) is satisfied. Conversely, suppose P is a Laurent polynomial as in (3.1.4), and such that (3.1.39) holds, according to which the right hand side of (3.1.41) is given by

$$\begin{bmatrix} P(1) \\ P(\alpha) \\ P(\alpha^2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{3.1.42}$$

Next, we observe that the determinant of the coefficient matrix in the left hand side of (3.1.41), as obtained from the co-factor expansion along the first row, and using (3.1.6), (3.1.7) and (3.1.5), is given by

$$\begin{aligned} \tfrac{1}{3}\left[(\alpha^4-\alpha^2)-(\alpha^2-\alpha)+(\alpha-\alpha^2)\right] = \alpha-\alpha^2 &= \alpha-(-\alpha-1) \\ &= 2\alpha+1 \\ &= 2e^{\frac{2\pi i}{3}}+1 \neq 0, \end{aligned}$$

according to which the coefficient matrix is invertible. It follows that the linear system (3.1.41), (3.1.42) has precisely one solution, which must therefore, by using (3.1.7) once again, be given by (2.2.8), and thereby completing our proof of the equivalence of (2.2.8) and (3.1.39).

Suppose now $\{p_j\}$ satisfies the sum-rule condition of order at least $m \in \mathbb{N}$, so that $\{p_j\}$ also satisfies the sum-rule condition (2.2.8), and its symbol P therefore satisfies (3.1.39). By applying (3.1.37) and (3.1.38), together with (3.1.11) and (3.1.7), we obtain, for $l = 1, \ldots, m-1$,

$$\begin{aligned} P^{(l)}(\alpha) = \tfrac{1}{3}\alpha^{2l}\Big[&\left(1+\alpha^2+\alpha\right)\beta_l + c_{l,0}\left(1+\alpha^2+\alpha\right)\beta_0 \\ &+\cdots+c_{l,l-1}\left(1+\alpha^2+\alpha\right)\beta_{l-1}\Big] = 0; \end{aligned}$$

$$\begin{aligned} P^{(l)}(\alpha^2) = \tfrac{1}{3}\alpha^{l}\Big[&\left(1+\alpha+\alpha^2\right)\beta_l + c_{l,0}\left(1+\alpha+\alpha^2\right)\beta_0 \\ &+\cdots+c_{l,l-1}\left(1+\alpha+\alpha^2\right)\beta_{l-1}\Big] = 0. \end{aligned}$$

and thus, by using also (3.1.39),

$$P^{(l)}(\alpha) = P^{(l)}(\alpha^2) = 0, \quad l = 0, \ldots, m-1. \tag{3.1.43}$$

Hence we deduce that there exists a Laurent polynomial R such that

$$\begin{aligned} P(z) &= \tfrac{1}{3^m}(z-\alpha)^m(z-\alpha^2)^m R(z) && (3.1.44) \\ &= \tfrac{1}{3^m}\left(z^2-(\alpha+\alpha^2)z+\alpha^3\right)^m R(z) \\ &= \left(\frac{z^2+z+1}{3}\right)^m R(z), && (3.1.45) \end{aligned}$$

from (3.1.6) and (3.1.7). Since, moreover, $P(1) = 1$ from (3.1.39), we have

$$R(1) = 1. \tag{3.1.46}$$

To prove the other direction, suppose that the Laurent polynomial P satisfies (3.1.31), (3.1.32). Our proof will be complete if we can show that the sequence $\{p_j\} \in l_0$ in (3.1.4) then satisfies

$$\sum_j (3j)^l p_{3j} = \sum_j (3j-1)^l p_{3j-1} = \sum_j (3j-2)^l p_{3j-2}, \quad l = 0, \ldots, m-1, \tag{3.1.47}$$

for then $\{p_j\}$ satisfies the sum-rule condition of order at least m.

First, observe from (3.1.31), (3.1.32), (3.1.6) and (3.1.7) that (3.1.39) holds, which is equivalent to the sum-rule condition (2.2.8), that is,

$$\sum_j p_{3j} = \sum_j p_{3j-1} = \sum_j p_{3j-2} = 1. \tag{3.1.48}$$

We proceed to show inductively that

$$\sum_j (3j)^l p_{3j} = \sum_j (3j-1)^l p_{3j-1} = \sum_j (3j-2)^l p_{3j-2},$$
$$l = 0, \ldots, k; \; k = 0, \ldots, m-1. \tag{3.1.49}$$

After noting from (3.1.48) that (3.1.49) holds for $k = 0$, suppose next that (3.1.49) holds for a fixed non-negative integer $k \leq m-2$. Our inductive proof of (3.1.49) will be complete if we can show that then

$$\sum_j (3j)^{k+1} p_{3j} = \sum_j (3j-1)^{k+1} p_{3j-1} = \sum_j (3j-2)^{k+1} p_{3j-2}. \tag{3.1.50}$$

To this end, we first apply the inductive assumption (3.1.49), together with (3.1.6) and (3.1.7), and the fact that (3.1.31), (3.1.6), (3.1.7) and $k+1 \leq m-1$ yield

$$P^{(k+1)}(\alpha) = P^{(k+1)}(\alpha^2) = 0,$$

to deduce from (3.1.37) and (3.1.38) that, respectively,

$$\left(\sum_j (3j)^{k+1} p_{3j} - \sum_j (3j-1)^{k+1} p_{3j-1} \right)$$
$$= \alpha \left(\sum_j (3j-1)^{k+1} p_{3j-1} - \sum_j (3j-2)^{k+1} p_{3j-2} \right); \tag{3.1.51}$$

$$\left(\sum_j (3j)^{k+1} p_{3j} - \sum_j (3j-2)^{k+1} p_{3j-2}\right)$$
$$= \alpha \left(\sum_j (3j-2)^{k+1} p_{3j-2} - \sum_j (3j-1)^{k+1} p_{3j-1}\right). \quad (3.1.52)$$

It then follows from (3.1.51) and (3.1.52), together with the fact that (3.1.5) implies $\mathrm{Im}(\alpha) \neq 0$, that (3.1.50) is indeed satisfied. Hence (3.1.49) holds, in which we may now set $k = m-1$ to obtain

$$\sum_j (3j)^l p_{3j} = \sum_j (3j-1)^l p_{3j-1} = \sum_j (3j-2)^l p_{3j-2}, \quad l = 0, \ldots, m-1. \tag{3.1.53}$$

■

It follows from Theorems 3.1.1, 3.1.12 and 3.1.4 that, if we require that the refinement sequence $\{p_j\}$ satisfies the sum-rule condition of order at least m for some $m \in \mathbb{N}$, then the necessary condition in Theorem 3.1.1 has the following equivalent formulation.

Corollary 3.1.13 *A sequence $\{p_j\} \in l_0$ satisfies the condition (3.1.2), as well as the sum-rule condition of order at least $m \in \mathbb{N}$, if and only if its corresponding three-scale symbol P is given by (3.1.31), where R is a Laurent polynomial satisfying the identity*

$$\left(\tfrac{1+z+z^2}{3}\right)^m R(z) + \left(\tfrac{1+\alpha z+\alpha^2 z^2}{3}\right)^m R(\alpha z) + \left(\tfrac{1+\alpha^2 z+\alpha z^2}{3}\right)^m R(\alpha^2 z) = 1, \quad z \in \mathbb{C}\backslash\{0\}, \tag{3.1.54}$$

as well as the condition

$$R(1) = 1, \tag{3.1.55}$$

and where the complex number α is defined by (3.1.5).

We proceed in Section 3.2 to explicitly construct a Laurent polynomial R of shortest possible length, satisfying (3.1.54), (3.1.55), and thereby yielding a minimally supported sequence $\{p_j\}$ satisfying the condition (3.1.2), as well as the sum-rule condition of order at least $m \in \mathbb{N}$.

3.2 Minimally supported refinement sequence

In this section, we shall obtain an explicit construction method for a polynomial U of minimum degree, satisfying the polynomial identity

$$\left(\tfrac{1+z+z^2}{3}\right)^m U(z) + \alpha^{2d_m} \left(\tfrac{1+\alpha z+\alpha^2 z^2}{3}\right)^m U(\alpha z) + \alpha^{d_m} \left(\tfrac{1+\alpha^2 z+\alpha z^2}{3}\right)^m U(\alpha^2 z) = z^{d_m},$$
$$z \in \mathbb{C} \setminus \{0\}, \quad (3.2.1)$$

as well as the condition

$$U(1) = 1, \tag{3.2.2}$$

where

$$d_m := \lfloor \tfrac{3m-1}{2} \rfloor, \tag{3.2.3}$$

for then the Laurent polynomial

$$R(z) := z^{-d_m} U(z) \tag{3.2.4}$$

yields a Laurent polynomial R of shortest possible length, satisfying the conditions (3.1.54), (3.1.55).

Our first result in this direction is the following.

Theorem 3.2.1 *For any $m \in \mathbb{N}$, there exists at most one polynomial solution $U \in \pi_{m-1}$ of the identity (3.2.1).*

Proof.

Let $U = \tilde{U}$ and $U = \tilde{\tilde{U}}$ be polynomial solutions in π_{m-1} of the identity (3.2.1). Our proof will be complete if we can show that $\tilde{U} = \tilde{\tilde{U}}$.

To this end, we define the polynomial

$$V := \tilde{U} - \tilde{\tilde{U}}, \tag{3.2.5}$$

according to which $V \in \pi_{m-1}$, with, moreover,

$$\begin{aligned}\left(1 + z + z^2\right)^m V(z) + \alpha^{2d_m} \left(1 + \alpha z + \alpha^2 z^2\right)^m V(\alpha z)&\\ + \alpha^{d_m} \left(1 + \alpha^2 z + \alpha z^2\right)^m V(\alpha^2 z) = 0, \quad z \in \mathbb{C}.&\end{aligned} \tag{3.2.6}$$

By setting $z = 1$ in (3.2.6), and using (3.1.7), we deduce that

$$V(1) = 0, \tag{3.2.7}$$

and thus

$$V(z) = (1 - z)\tilde{V}(z), \tag{3.2.8}$$

for some polynomial $\tilde{V} \in \pi_{m-2}$, so that also, from (3.1.6),

$$V(\alpha z) = \alpha(\alpha^2 - z)\tilde{V}(\alpha z); \quad V(\alpha^2 z) = \alpha^2(\alpha - z)\tilde{V}(\alpha^2 z). \tag{3.2.9}$$

By substituting (3.2.8) and (3.2.9) into (3.2.6), and using (3.1.6) and (3.1.7), we obtain the identity

$$(1-z^3)\Big[\left(1+z+z^2\right)^{m-1}\tilde{V}(z)+\alpha^{2d_m}\left(1+\alpha z+\alpha^2 z^2\right)^{m-1}\tilde{V}(\alpha z)$$
$$+\alpha^{d_m}\left(1+\alpha^2 z+\alpha z^2\right)^{m-1}\tilde{V}(\alpha^2 z)\Big]=0,\quad z\in\mathbb{C},\quad (3.2.10)$$

according to which

$$\left(1+z+z^2\right)^{m-1}\tilde{V}(z)+\alpha^{2d_m}\left(1+\alpha z+\alpha^2 z^2\right)^{m-1}\tilde{V}(\alpha z)$$
$$+\alpha^{d_m}\left(1+\alpha^2 z+\alpha z^2\right)^{m-1}\tilde{V}(\alpha^2 z)=0,\quad z\in\mathbb{C}.\quad (3.2.11)$$

As in the derivation of (3.2.8) from (3.2.6), we deduce from (3.2.11) that

$$\tilde{V}(z)=(1-z)\tilde{\tilde{V}}(z) \quad (3.2.12)$$

for some polynomial $\tilde{\tilde{V}}\in\pi_{m-3}$, which, together with (3.2.8), yields

$$V(z)=(1-z)^2\tilde{\tilde{V}}(z), \quad (3.2.13)$$

where $\tilde{\tilde{V}}$ satisfies the identity

$$\left(1+z+z^2\right)^{m-2}\tilde{\tilde{V}}(z)+\alpha^{2d_m}\left(1+\alpha z+\alpha^2 z^2\right)^{m-2}\tilde{\tilde{V}}(\alpha z)$$
$$+\alpha^{d_m}\left(1+\alpha^2 z+\alpha z^2\right)^{m-2}\tilde{\tilde{V}}(\alpha^2 z)=0,\quad z\in\mathbb{C}.\quad (3.2.14)$$

Repeated applications of the above procedure yields the existence of a constant polynomial $V^*\in\pi_0$ such that

$$V(z)=(1-z)^{m-1}V^*(z), \quad (3.2.15)$$

and

$$\left(1+z+z^2\right)V^*(z)+\alpha^{2d_m}\left(1+\alpha z+\alpha^2 z^2\right)V^*(\alpha z)$$
$$+\alpha^{d_m}\left(1+\alpha^2 z+\alpha z^2\right)V^*(\alpha^2 z)=0,\quad z\in\mathbb{C}.\quad (3.2.16)$$

By setting $z=1$ in (3.2.16) and using (3.1.7), we deduce that

$$V^*(1)=0,$$

and thus, since $V^*\in\pi_0$, V^* is the zero polynomial. Hence, from (3.2.5) and (3.2.15), we have $\tilde{U}=\tilde{\tilde{U}}$, which is the required uniqueness result. ■

It follows from Theorem 3.2.1 that a polynomial solution $U\in\pi_{m-1}$ of the identity (3.2.1) is the minimum degree polynomial that satisfies (3.2.1). Indeed, suppose that there exists a polynomial $\tilde{U}\in\pi_n$, with $n<m-1$, which also satisfies (3.2.1). But, since $\pi_n\subset\pi_{m-1}$ for $n<m-1$, we must have $\tilde{U}\in\pi_{m-1}$, which implies that the polynomial solution U is not the unique

polynomial in π_{m-1} which satisfies (3.2.1), and thereby contradicting Theorem 3.2.1.

Based on Theorem 3.2.1, we proceed to investigate the existence and explicit construction of a polynomial solution $U \in \pi_{m-1}$ of the identity (3.2.1). To this end, for any $U \in \pi_{m-1}$, define

$$\begin{aligned} Q(z) := \left(\tfrac{1+z+z^2}{3}\right)^m U(z) + \alpha^{2d_m}\left(\tfrac{1+\alpha z+\alpha^2 z^2}{3}\right)^m U(\alpha z) \\ + \alpha^{d_m}\left(\tfrac{1+\alpha^2 z+\alpha z^2}{3}\right)^m U(\alpha^2 z) - z^{d_m}, \quad z \in \mathbb{C}, \quad (3.2.17) \end{aligned}$$

according to which the identity (3.2.1) has the equivalent formulation

$$Q(z) = 0, \quad z \in \mathbb{C}. \tag{3.2.18}$$

Since $U \in \pi_{m-1}$, it follows from (3.2.17) and (3.2.3) that

$$Q \in \pi_{3m-1}. \tag{3.2.19}$$

We deduce from (3.2.17) and (3.2.18) that a solution $U \in \pi_{m-1}$ of the identity (3.2.1) must be such that

$$Q^{(n)}(1) = 0, \quad n = 0, \ldots, m-1, \tag{3.2.20}$$

or equivalently, by using the differentiation formula

$$\tfrac{d^n}{dz^n}(z^l) = n!\tbinom{l}{n} z^{l-n}, \quad n = 0, \ldots, l, \tag{3.2.21}$$

for each $l \in \mathbb{N}$, together with (3.1.7),

$$\tfrac{d^n}{dz^n}\left[\left(\tfrac{1+z+z^2}{3}\right)^m U(z)\right]\Bigg|_{z=1} = \tfrac{d^n}{dz^n}(z^{d_m})\Bigg|_{z=1} = n!\tbinom{d_m}{n}, \quad n = 0, \ldots, m-1. \tag{3.2.22}$$

Note that the case $n = 0$ in (3.2.22) yields

$$U(1) = 1. \tag{3.2.23}$$

By applying the Leibniz rule for differentiation twice, and using (3.2.21), we obtain, for $n = 1, \ldots, m-1$,

$$\begin{aligned} &\tfrac{d^n}{dz^n}\left[\left(\tfrac{1+z+z^2}{3}\right)^m U(z)\right] \\ &= \tfrac{1}{3^m}\sum_{l=0}^{n}\tbinom{n}{l}\left[\tfrac{d^l}{dz^l}(1+z+z^2)^m\right]U^{(n-l)}(z) \\ &= \tfrac{1}{3^m}\sum_{l=0}^{n}\tbinom{n}{l}\left[\tfrac{d^l}{dz^l}\left(\sum_{j=0}^{m}\tbinom{m}{j}(1+z)^{m-j}z^{2j}\right)\right]U^{(n-l)}(z) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3^m}\sum_{l=0}^{n}\binom{n}{l}\sum_{j=0}^{m}\binom{m}{j}\left[\frac{d^l}{dz^l}\left((1+z)^{m-j}z^{2j}\right)\right]U^{(n-l)}(z)\\
&= \frac{1}{3^m}\sum_{l=0}^{n}\binom{n}{l}\sum_{j=0}^{m}\binom{m}{j}\left[\sum_{k=0}^{l}\binom{l}{k}\left(\frac{d^k}{dz^k}(1+z)^{m-j}\right)\left(\frac{d^{l-k}}{dz^{l-k}}(z^{2j})\right)\right]U^{(n-l)}(z)\\
&= \frac{1}{3^m}\sum_{l=0}^{n}\binom{n}{l}\left[\sum_{j=0}^{m}\binom{m}{j}\sum_{k=0}^{l}\binom{l}{k}\left(k!\binom{m-j}{k}(1+z)^{m-j-k}\right)\left((l-k)!\binom{2j}{l-k}z^{2j-l+k}\right)\right]\\
&\qquad\times\ U^{(n-l)}(z)\\
&= \frac{1}{3^m}\sum_{l=0}^{n}l!\binom{n}{l}\left[\sum_{j=0}^{m}\binom{m}{j}\sum_{k=0}^{l}\binom{m-j}{k}\binom{2j}{l-k}(1+z)^{m-j-k}z^{2j-l+k}\right]U^{(n-l)}(z),
\end{aligned}
\tag{3.2.24}
$$

and thus

$$
\frac{d^n}{dz^n}\left[\left(\frac{1+z+z^2}{3}\right)^m U(z)\right]\Bigg|_{z=1} = \sum_{l=0}^{n}\gamma_{m,n,l}U^{(n-l)}(1), \tag{3.2.25}
$$

where

$$
\gamma_{m,n,l} := \left(\frac{1}{3}\right)^m l!\binom{n}{l}\sum_{j=0}^{m}\binom{m}{j}\left[\sum_{k=0}^{l}\binom{m-j}{k}\binom{2j}{l-k}2^{m-j-k}\right],
$$
$$
l=0,\ldots,n;\quad n=1,\ldots,m-1. \tag{3.2.26}
$$

It follows from (3.2.22), (3.2.23) and (3.2.25) that the condition (3.2.20) can equivalently be formulated, in terms of the polynomial U, as

$$
\left.\begin{aligned}
U(1) &= 1;\\
\sum_{l=0}^{n}\gamma_{m,n,l}U^{(n-l)}(1) &= n!\binom{d_m}{n},\quad n=1,\ldots,m-1,
\end{aligned}\right\} \tag{3.2.27}
$$

where the sequence $\{\gamma_{m,n,l} : l=0,\ldots,n;\ n=1,\ldots,m-1\}$ is given by (3.2.26). By observing from (3.2.26) that

$$
\gamma_{m,n,0} = \frac{1}{3^m}\sum_{j=0}^{m}\binom{m}{j}2^{m-j} = \frac{1}{3^m}(3^m) = 1,
$$

it follows that (3.2.27) has the equivalent formulation

$$
\left.\begin{aligned}
U(1) &= 1;\\
U^{(n)}(1) &= n!\binom{d_m}{n} - \sum_{l=1}^{n}\gamma_{m,n,l}U^{(n-l)}(1),\quad n=1,\ldots,m-1.
\end{aligned}\right\} \tag{3.2.28}
$$

Hence we have shown that a solution $U \in \pi_{m-1}$ of the identity (3.2.1) must be such that the sequence $\{U^{(n)}(1) : n = 0, \ldots, m-1\}$ satisfies the recursive formulation (3.2.28). Thus, with the polynomial $U_m \in \pi_{m-1}$ defined by

$$U_m(z) := \sum_{j=0}^{m-1} \frac{\beta_{m,j}}{j!}(z-1)^j, \tag{3.2.29}$$

where the sequence $\{\beta_{m,j} : j = 0, \ldots, m-1\}$ is given recursively by

$$\left.\begin{aligned} \beta_{m,0} &= 1; \\ \beta_{m,j} &= j!\tbinom{d_m}{j} - \sum_{k=1}^{j} \gamma_{m,j,k}\beta_{m,j-k}, \quad j = 1, \ldots, m-1, \end{aligned}\right\} \tag{3.2.30}$$

and with the sequence $\{\gamma_{m,j,k} : k = 0, \ldots, j;\ j = 1, \ldots, m-1\}$ given as in (3.2.26), it follows that a polynomial $U \in \pi_{m-1}$ is a solution of the identity (3.2.1) only if $U = U_m$.

We proceed to prove that $U = U_m$ does indeed satisfy the identity (3.2.1). From the equivalence of (3.2.1) and (3.2.18), it will suffice to prove that the polynomial

$$\begin{aligned} Q_m(z) := \left(\tfrac{1+z+z^2}{3}\right)^m U_m(z) + \alpha^{2d_m}\left(\tfrac{1+\alpha z+\alpha^2 z^2}{3}\right)^m U_m(\alpha z) \\ + \alpha^{d_m}\left(\tfrac{1+\alpha^2 z+\alpha z^2}{3}\right)^m U_m(\alpha^2 z) - z^{d_m}, \quad z \in \mathbb{C}, \end{aligned} \tag{3.2.31}$$

satisfies

$$Q_m(z) = 0, \quad z \in \mathbb{C}. \tag{3.2.32}$$

To this end, we first observe from the equivalence of (3.2.20) and (3.2.28), together with (3.2.29), (3.2.30), that

$$Q_m^{(n)}(1) = 0, \quad n = 0, \ldots, m-1. \tag{3.2.33}$$

Next, we note from (3.2.31) and (3.1.6) that

$$\begin{aligned} Q_m(\alpha z) &= \left(\tfrac{1+\alpha z+\alpha^2 z^2}{3}\right)^m U_m(\alpha z) + \alpha^{2d_m}\left(\tfrac{1+\alpha^2 z+\alpha z^2}{3}\right)^m U_m(\alpha^2 z) \\ &\quad + \alpha^{d_m}\left(\tfrac{1+z+z^2}{3}\right)^m U_m(z) - \alpha^{d_m} z^{d_m} \\ &= \alpha^{d_m}\left[\left(\tfrac{1+z+z^2}{3}\right)^m U_m(z) + \alpha^{2d_m}\left(\tfrac{1+\alpha z+\alpha^2 z^2}{3}\right)^m U_m(\alpha z)\right. \\ &\quad \left. + \alpha^{d_m}\left(\tfrac{1+\alpha^2 z+\alpha z^2}{3}\right)^m U_m(\alpha^2 z) - z^{d_m}\right] \\ &= \alpha^{d_m} Q_m(z), \end{aligned}$$

and thus

$$Q_m(z) = \alpha^{2d_m} Q_m(\alpha z),$$

according to which

$$Q_m^{(n)}(z) = \alpha^{2d_m+n} Q_m^{(n)}(\alpha z), \quad n = 0, \ldots, m-1,$$

which, together with (3.2.33), yields

$$Q_m^{(n)}(\alpha) = \alpha^{d_m-n} Q_m^{(n)}(1) = 0, \quad n = 0, \ldots, m-1. \tag{3.2.34}$$

Similarly, it follows from (3.2.31) and (3.1.6) that

$$\begin{aligned}
Q_m(\alpha^2 z) &= \left(\tfrac{1+\alpha^2 z+\alpha z^2}{3}\right)^m U_m(\alpha^2 z) + \alpha^{2d_m}\left(\tfrac{1+z+z^2}{3}\right)^m U_m(z) \\
&\quad + \alpha^{d_m}\left(\tfrac{1+\alpha z+\alpha^2 z^2}{3}\right)^m U_m(\alpha z) - \alpha^{2d_m} z^{d_m} \\
&= \alpha^{2d_m}\left[\left(\tfrac{1+z+z^2}{3}\right)^m U_m(z) + \alpha^{2d_m}\left(\tfrac{1+\alpha z+\alpha^2 z^2}{3}\right)^m U_m(\alpha z)\right. \\
&\quad \left. +\alpha^{d_m}\left(\tfrac{1+\alpha^2 z+\alpha z^2}{3}\right)^m U_m(\alpha^2 z) - z^{d_m}\right] \\
&= \alpha^{2d_m} Q_m(z),
\end{aligned}$$

and thus

$$Q_m(z) = \alpha^{d_m} Q_m(\alpha^2 z),$$

according to which

$$Q_m^{(n)}(z) = \alpha^{d_m+2n} Q_m^{(n)}(\alpha^2 z), \quad n = 0, \ldots, m-1,$$

which, together with (3.2.33), yields

$$Q_m^{(n)}(\alpha^2) = \alpha^{2d_m-2n} Q_m^{(n)}(1) = 0, \quad n = 0, \ldots, m-1. \tag{3.2.35}$$

It follows from (3.2.33), (3.2.34) and (3.2.35) that the polynomial Q_m satisfies the formulation

$$Q_m(z) = (z-1)^m (z-\alpha)^m (z-\alpha^2)^m \tilde{Q}_m(z) \tag{3.2.36}$$

for some polynomial $\tilde{Q}_m$. But, as in (3.2.19), we have

$$Q_m \in \pi_{3m-1}. \tag{3.2.37}$$

It follows from (3.2.36) and (3.2.37) that $\tilde{Q}_m$ is the zero polynomial, which, together with (3.2.36), yields the desired result (3.2.32).

By recalling also the uniqueness result of Theorem 3.2.1, we have therefore established the following result.

Theorem 3.2.2 *The polynomial $U = U_m \in \pi_{m-1}$, as defined by (3.2.29), (3.2.30), (3.2.26), is the polynomial of minimum degree satisfying both the identity (3.2.1) and the condition (3.2.2).*

Observe that the computation of the polynomial U_m by means of (3.2.29), (3.2.30) and (3.2.26) grows rapidly in intensity as m increases. In Section 3.4, we shall derive a considerably more efficient recursive formulation for the computation of U_m.

By combining Corollary 3.1.13, Theorem 3.2.2 and (3.2.4), we obtain the following result.

Corollary 3.2.3 *A sequence $\{p_j\} = \{p_{m,j}\} \in l_0$ is the minimally supported sequence satisfying the condition (3.1.2), as well as the sum-rule condition of order at least $m \in \mathbb{N}$, if and only if its corresponding three-scale symbol $P = P_m$ is given by*

$$P_m(z) = \frac{1}{3}\sum_j p_{m,j} z^j = \left(\frac{1+z+z^2}{3}\right)^m z^{-d_m} U_m(z), \tag{3.2.38}$$

with U_m defined by (3.2.29), (3.2.30), (3.2.26), and d_m given by (3.2.3).

In the following section, we will derive some properties of this refinement sequence $\{p_{m,j}\}$.

3.3 Properties of refinement sequences

We begin by defining the concept of symmetry in polynomials.

Definition 3.3.1 *For $k = 0, 1, \ldots,$ a polynomial $f \in \pi_k$ is said to be a symmetric polynomial if*

$$z^k f\left(\tfrac{1}{z}\right) = f(z), \quad z \in \mathbb{C} \setminus \{0\}. \tag{3.3.1}$$

Observe that, if

$$f(z) = \sum_{j=0}^{k} c_j z^j,$$

then (3.3.1) is equivalent to the condition

$$c_{k-j} = c_j, \quad j = 0, \ldots, k.$$

Note that the polynomial

$$\tilde{P}_m(z) := \left(\frac{1+z+z^2}{3}\right)^m \tag{3.3.2}$$

is symmetric, since, for any $z \in \mathbb{C} \setminus \{0\}$, we have

$$\begin{aligned} z^{2m}\tilde{P}_m\left(z^{-1}\right) &= z^{2m}\left(\frac{1+z^{-1}+z^{-2}}{3}\right)^m \\ &= z^{2m}\left(\tfrac{1}{3}\right)^m\left(\frac{z^2+z+1}{z^2}\right)^m \\ &= z^{2m-2m}\left(\tfrac{1}{3}\right)^m\left(z^2+z+1\right)^m \\ &= \left(\frac{1+z+z^2}{3}\right)^m = \tilde{P}_m(z). \end{aligned} \tag{3.3.3}$$

We will rely on the following properties of the polynomial U_m of Theorem 3.2.2.

Theorem 3.3.2 *Let the polynomial U_m be defined by (3.2.29), (3.2.30), (3.2.26). Then the following hold:*

(i)

$$U_m \in \begin{cases} \pi_{m-1} & \textit{if } m \textit{ is odd;} \\ \pi_{m-2} & \textit{if } m \textit{ is even.} \end{cases} \tag{3.3.4}$$

(ii) U_m is a symmetric polynomial, that is,

$$U_m(z) = \begin{cases} z^{m-1}U_m\left(z^{-1}\right) & \textit{if } m \textit{ is odd;} \\ z^{m-2}U_m\left(z^{-1}\right) & \textit{if } m \textit{ is even.} \end{cases} \tag{3.3.5}$$

Proof.

(i) By the construction of U_m, given by (3.2.29), (3.2.30), (3.2.26), we know that $U_m \in \pi_{m-1}$. It therefore remains to prove that if m is even, then $U_m \in \pi_{m-2}$. To this end, let

$$m = 2n, \quad n \in \mathbb{N}, \tag{3.3.6}$$

so that $U_m = U_{2n} \in \pi_{2n-1}$, with $d_m = d_{2n} = \lfloor \frac{6n-1}{2} \rfloor = 3n-1$, from (3.2.3), and $\tilde{P}_m = \tilde{P}_{2n} \in \pi_{4n}$, with $\tilde{P}_m$ given by (3.3.2). Suppose $\{u_{2n,j} : j = 0, \ldots, 2n-1\}$ and $\{\tilde{p}_{2n,j} : j = 0, \ldots, 4n\}$ are coefficient sequences such that

$$U_{2n}(z) = \sum_{j=0}^{2n-1} u_{2n,j} z^j; \quad \tilde{P}_{2n}(z) = \sum_{j=0}^{4n} \tilde{p}_{2n,j} z^j. \tag{3.3.7}$$

Now substitute (3.3.7) in (3.2.1) to obtain, by using also the definition (3.3.2),

$$\left(\tilde{p}_{2n,4n}u_{2n,2n-1}\right)z^{6n-1}+\left(\alpha^{12n-3}\tilde{p}_{2n,4n}u_{2n,2n-1}\right)z^{6n-1}$$
$$+\left(\alpha^{15n-3}\tilde{p}_{2n,4n}u_{2n,2n-1}\right)z^{6n-1}+V(z)=z^{3n-1}, \quad (3.3.8)$$

for some $V\in\pi_{6n-2}$, from which it follows, by recalling also (3.1.6), that

$$\left(3\tilde{p}_{2n,4n}u_{2n,2n-1}\right)z^{6n-1}+V(z)=z^{3n-1}. \quad (3.3.9)$$

By comparing the degrees of the left hand and right hand sides of (3.3.9), we deduce that

$$\tilde{p}_{2n,4n}u_{2n,2n-1}=0, \quad (3.3.10)$$

and thus, since $\tilde{p}_{2n,4n}\neq 0$, from (3.3.2),

$$u_{2n,2n-1}=0, \quad (3.3.11)$$

that is, $U_{2n}\in\pi_{2n-2}$.

(ii) We prove (3.3.5) for the case where m is odd; the proof of the case where m is even, follows similarly. Let m be given by

$$m=2n-1, \quad n\in\mathbb{N},$$

so that $U_m=U_{2n-1}\in\pi_{2n-2}$, from (i), with $d_m=d_{2n-1}=\lfloor\frac{6n-4}{2}\rfloor=3n-2$, from (3.2.3). Define the polynomial $\tilde{P}_m=\tilde{P}_{2n-1}\in\pi_{4n-2}$ by (3.3.2). By replacing z by z^{-1} in (3.2.1), and multiplying with $z^{2m}z^{m-1}=z^{4n-2}z^{2n-2}$, we obtain, by using also the definition (3.3.2),

$$z^{4n-2}\tilde{P}_{2n-1}\left(z^{-1}\right)z^{2n-2}U_{2n-1}\left(z^{-1}\right)$$
$$+\alpha^{6n-4}z^{4n-2}\tilde{P}_{2n-1}\left(\alpha z^{-1}\right)z^{2n-2}U_{2n-1}\left(\alpha z^{-1}\right)$$
$$+\alpha^{3n-2}z^{4n-2}\tilde{P}_{2n-1}\left(\alpha^{2}z^{-1}\right)z^{2n-2}U_{2n-1}\left(\alpha^{2}z^{-1}\right)=z^{3n-2}.$$

Since $\tilde{P}_{2n-1}$ is a symmetric polynomial, we may apply (3.3.3) with $m=2n-1$ to obtain

$$\tilde{P}_{2n-1}(z)z^{2n-2}U_{2n-1}\left(z^{-1}\right)+\alpha^{6n-4}\tilde{P}_{2n-1}(\alpha z)z^{2n-2}U_{2n-1}\left(\alpha z^{-1}\right)$$
$$+\alpha^{3n-2}\tilde{P}_{2n-1}(\alpha^{2}z)z^{2n-2}U_{2n-1}\left(\alpha^{2}z^{-1}\right)=z^{3n-2}. \quad (3.3.12)$$

With the definition

$$U_m^*(z)=U_{2n-1}^*:=z^{2n-2}U_{2n-1}\left(z^{-1}\right)\in\pi_{2n-2}, \quad (3.3.13)$$

(3.3.12) becomes

$$\tilde{P}_{2n-1}(z)U_{2n-1}^*(z)+\alpha^{6n-4}\tilde{P}_{2n-1}(\alpha z)U_{2n-1}^*(\alpha z)$$
$$+\alpha^{3n-2}\tilde{P}_{2n-1}(\alpha^{2}z)U_{2n-1}^*(\alpha^{2}z)=z^{3n-2},$$

that is, $U^*_{2n-1} \in \pi_{2n-2}$ satisfies (3.2.1). However, we have shown in Theorem 3.2.1 that U_{2n-1} is the unique polynomial in π_{2n-2} that satisfies (3.2.1). We therefore deduce that $U_{2n-1} = U^*_{2n-1}$, that is, from (3.3.13),

$$U_{2n-1}(z) = z^{2n-2} U_{2n-1}\left(z^{-1}\right),$$

so that U_{2n-1} is indeed a symmetric polynomial. ■

By using Theorem 3.3.2, we prove the following properties of the sequence $\{p_{m,j}\}$, given by (3.2.38) in Corollary 3.2.3.

Theorem 3.3.3 *For $m \in \mathbb{N}$, the sequence $\{p_{m,j}\}$, defined by (3.2.38) in Corollary 3.2.3, satisfies the following properties:*

(i)

$$p_{m,3j} = \delta_j, \quad j \in \mathbb{Z}; \tag{3.3.14}$$

(ii)

$$\text{supp}\,\{p_{m,j}\} \subseteq \left[-\lfloor \tfrac{3m-1}{2} \rfloor, \lfloor \tfrac{3m-1}{2} \rfloor\right]\Big|_{\mathbb{Z}}; \tag{3.3.15}$$

(iii) $\{p_{m,j}\}$ is a symmetric sequence.

Proof.

(i) The result follows immediately from Corollary 3.2.3.

(ii) First, let m be odd, with

$$m = 2n-1, \quad n \in \mathbb{N},$$

so that $\deg(U_m) = \deg(U_{2n-1}) \leq 2n-2$, from (3.3.4) in Theorem 3.3.2, and $d_m = d_{2n-1} = \lfloor \frac{6n-4}{2} \rfloor = 3n-2$, from (3.2.3). It follows from (3.2.38) in Corollary 3.2.3, with $m = 2n-1$ and $d_{2n-1} = 3n-2$, that

$$\sum_j p_{2n-1,j} z^j = \left(\tfrac{1}{3}\right)^{2n-2} \left(1+z+z^2\right)^{2n-1} z^{-3n+2} U_{2n-1}(z),$$

so that, by using also (3.2.3),

$$\begin{aligned} \text{supp}\,\{p_{2n-1,j}\} \subseteq [-(3n-2),(3n-2)]\big|_{\mathbb{Z}} &= [-d_{2n-1}, d_{2n-1}]\big|_{\mathbb{Z}} \\ &= [-d_m, d_m]\big|_{\mathbb{Z}} \\ &= \left[-\lfloor \tfrac{3m-1}{2} \rfloor, \lfloor \tfrac{3m-1}{2} \rfloor\right]\Big|_{\mathbb{Z}}. \end{aligned}$$

Next, let m be even, with

$$m = 2n, \quad n \in \mathbb{N},$$

so that $\deg(U_m) = \deg(U_{2n}) \leq 2n-2$, from (3.3.4) in Theorem 3.3.2, with $d_m = d_{2n} = \lfloor \frac{6n-1}{2} \rfloor = 3n-1$, from (3.2.3). It follows from (3.2.38) in Corollary 3.2.3, with $m = 2n$ and $d_{2n} = 3n-1$, that

$$\sum_j p_{2n,j} z^j = \left(\tfrac{1}{3}\right)^{2n-1} \left(1+z+z^2\right)^{2n} z^{-3n+1} U_{2n}(z),$$

so that, by using also (3.2.3),

$$\begin{aligned} \operatorname{supp}\{p_{2n,j}\} \subseteq [-(3n-1),(3n-1)]\big|_{\mathbb{Z}} &= [-d_{2n}, d_{2n}]\big|_{\mathbb{Z}} \\ &= [-d_m, d_m]\big|_{\mathbb{Z}} \\ &= \left[-\lfloor \tfrac{3m-1}{2} \rfloor, \lfloor \tfrac{3m-1}{2} \rfloor\right]\Big|_{\mathbb{Z}}, \quad (3.3.16) \end{aligned}$$

and thereby completing our proof.

(iii) The result follows immediately from (3.2.38) in Corollary 3.2.3, since $\tilde{P}_m(z) = \left(\frac{1+z+z^2}{3}\right)^m$ and U_m are symmetric polynomials, as seen, respectively, in (3.3.3) and (3.3.5). ■

It is interesting to note that the Dubuc-Deslauries interpolatory 2-refinement sequence $\{p_j\}$, analogous to the 3-refinement sequence in Corollary 3.2.3, is only symmetric for even values of m (see [1], Theorem 8.2.2, p 304).

3.4 A recursive formulation

In this section, we derive a useful recursion formula for the efficient computation of the polynomial U_m of Theorem 3.2.2.

To this end, we start by observing that $U_1(z) = 1$, from (3.3.4) and the fact that $U_m(1) = 1$ for any $m \in \mathbb{N}$, as in (3.2.28). Next, we note, by successively setting $m = 2n-1$ and $m = 2n$ in (3.2.1), for some $n \in \mathbb{N}$, and subtracting the resulting identities, and using also (3.3.2) and the fact that (3.2.3) yields $d_{2n-1} = 3n-2$ and $d_{2n} = 3n-1$, that

$$\begin{aligned} &\tilde{P}_{2n-1}(z)\left[\tilde{P}_1(z)U_{2n}(z) - zU_{2n-1}(z)\right] \\ &\qquad + \alpha\tilde{P}_{2n-1}(\alpha z)\left[\tilde{P}_1(\alpha z)U_{2n}(\alpha z) - (\alpha z)U_{2n-1}(\alpha z)\right] \\ &\quad + \alpha^2\tilde{P}_{2n-1}(\alpha^2 z)\left[\tilde{P}_1(\alpha^2 z)U_{2n}(\alpha^2 z) - (\alpha^2 z)U_{2n-1}(\alpha^2 z)\right] = 0. \quad (3.4.1) \end{aligned}$$

With the definition

$$V_{2n-1}(z) := \tilde{P}_1(z)U_{2n}(z) - zU_{2n-1}(z), \qquad (3.4.2)$$

(3.4.1) becomes

$$\tilde{P}_{2n-1}(z)V_{2n-1}(z) + \alpha\tilde{P}_{2n-1}(\alpha z)V_{2n-1}(\alpha z) + \alpha^2\tilde{P}_{2n-1}(\alpha^2 z)V_{2n-1}(\alpha^2 z) = 0. \tag{3.4.3}$$

Observe, from (3.3.2) and (3.3.4), that $V_{2n-1} \in \pi_{2n}$. Next, we set $z = 1$ in (3.4.3) to obtain, by using also the fact that (3.3.2), (3.1.6) and (3.1.7) yield $\tilde{P}_{2n-1}(1) = 1$ and $\tilde{P}_{2n-1}(\alpha) = \tilde{P}_{2n-1}(\alpha^2) = 0$,

$$V_{2n-1}(1) = 0,$$

from which we deduce that

$$V_{2n-1}(z) = (1-z)V_{2n-2}(z),$$

for some polynomial $V_{2n-2} \in \pi_{2n-1}$, so that (3.4.3) becomes

$$\begin{aligned}\tilde{P}_{2n-1}(z)(1-z)V_{2n-2}(z) + \alpha\tilde{P}_{2n-1}(\alpha z)(1-\alpha z)V_{2n-2}(\alpha z)&\\ + \alpha^2\tilde{P}_{2n-1}(\alpha^2 z)(1-\alpha^2 z)V_{2n-2}(\alpha^2 z) = 0.&\end{aligned} \tag{3.4.4}$$

We observe, by using (3.3.2), that

$$\begin{aligned}\tilde{P}_{2n-1}(z)(1-z) &= \tilde{P}_{2n-2}(z)\tilde{P}_1(z)(1-z) = \tfrac{1}{3}\tilde{P}_{2n-2}(z)(1-z^3);\\ \tilde{P}_{2n-1}(\alpha z)(1-\alpha z) &= \tilde{P}_{2n-2}(\alpha z)\tilde{P}_1(\alpha z)(1-\alpha z) = \tfrac{1}{3}\tilde{P}_{2n-2}(\alpha z)(1-z^3);\\ \tilde{P}_{2n-1}(\alpha^2 z)(1-\alpha^2 z) &= \tilde{P}_{2n-2}(\alpha^2 z)\tilde{P}_1(\alpha^2 z)(1-\alpha^2 z) = \tfrac{1}{3}\tilde{P}_{2n-2}(\alpha^2 z)(1-z^3),\end{aligned}$$

so that it follows from (3.4.4) that

$$\tilde{P}_{2n-2}(z)V_{2n-2}(z) + \alpha\tilde{P}_{2n-2}(\alpha z)V_{2n-2}(\alpha z) + \alpha^2\tilde{P}_{2n-2}(\alpha^2 z)V_{2n-2}(\alpha^2 z) = 0. \tag{3.4.5}$$

Repeated applications of this procedure eventually yields a polynomial $V_0 \in \pi_1$ such that

$$V_0(z) + \alpha V_0(\alpha z) + \alpha^2 V_0(\alpha^2 z) = 0, \tag{3.4.6}$$

and with

$$V_{2n-1}(z) = (1-z)^{2n-1}V_0(z). \tag{3.4.7}$$

Next, we note, from the symmetry properties of the polynomials $\tilde{P}_1$, U_{2n} and U_{2n-1}, as given, respectively, in (3.3.3) and (3.3.5), that, for $z \in \mathbb{C} \setminus \{0\}$,

$$\tilde{P}_1(z^{-1}) = z^{-2}\tilde{P}_1(z); \quad U_{2n}(z^{-1}) = z^{-2n+2}U_{2n}(z); \quad U_{2n-1}(z^{-1}) = z^{-2n+2}U_{2n-1}(z),$$

and thus, from (3.4.2),

$$\begin{aligned}V_{2n-1}(z^{-1}) &= \tilde{P}_1(z^{-1})U_{2n}(z^{-1}) - z^{-1}U_{2n-1}(z^{-1})\\ &= z^{-2}\tilde{P}_1(z)z^{-2n+2}U_{2n}(z) - z^{-1}z^{-2n+2}U_{2n-1}(z)\end{aligned}$$

$$= z^{-2n}\left[\tilde{P}_1(z)U_{2n}(z) - zU_{2n-1}(z)\right] = z^{-2n}V_{2n-1}(z). \tag{3.4.8}$$

By combining (3.4.8) and (3.4.7), we obtain

$$\left(1 - z^{-1}\right)^{2n-1} V_0(z^{-1}) = z^{-2n}(1-z)^{2n-1}V_0(z), \quad z \in \mathbb{C} \setminus \{0\},$$

that is,

$$V_0(z^{-1}) = -z^{-1}V_0(z), \quad z \in \mathbb{C} \setminus \{0\}. \tag{3.4.9}$$

But $V_0 \in \pi_1$, so that

$$V_0(z) = az + b, \tag{3.4.10}$$

for some $a, b \in \mathbb{R}$, which, together with (3.4.9), yields

$$\tfrac{a}{z} + b = -a - \tfrac{b}{z}, \quad z \in \mathbb{C} \setminus \{0\}.$$

This implies $b = -a$, so that, from (3.4.10),

$$V_0(z) = K_1(1-z),$$

for some $K_1 \in \mathbb{R}$, and thus, using also (3.4.7),

$$V_{2n-1}(z) = K_1(1-z)^{2n}. \tag{3.4.11}$$

It follows from (3.4.2) and (3.4.11) that

$$\tilde{P}_1(z)U_{2n}(z) - zU_{2n-1}(z) = K_1(1-z)^{2n}, \tag{3.4.12}$$

in which we now set $z = \alpha$ to obtain, by using also (3.3.2) and (3.1.7),

$$K_1 = -\alpha U_{2n-1}(\alpha)\left[(1-\alpha)^2\right]^{-n} = \frac{(-1)^{n+1}}{3^n}\alpha^{1-n}U_{2n-1}(\alpha). \tag{3.4.13}$$

By combining (3.4.12) and (3.4.13), and using also the definition (3.3.2), we obtain, for $z \in \mathbb{C} \setminus \{\alpha, \alpha^2\}$, the recursion formulation

$$U_{2n}(z) = \frac{3zU_{2n-1}(z) + \frac{(-1)^{n+1}}{3^{n-1}}\alpha^{1-n}U_{2n-1}(\alpha)(1-z)^{2n}}{1 + z + z^2}, \quad n \in \mathbb{N}. \tag{3.4.14}$$

We proceed to derive a similar recursion formulation for the computation of U_{2n+1}. To this end, we successively set $m = 2n$ and $m = 2n+1$ in (3.2.1), for some $n \in \mathbb{N}$, and subtract the resulting identities to obtain, using also (3.3.2) and the fact that (3.2.3) yields $d_{2n} = 3n - 1$ and $d_{2n+1} = 3n + 1$,

$$\tilde{P}_{2n}(z)\left[\tilde{P}_1(z)U_{2n+1}(z) - z^2U_{2n}(z)\right]$$
$$+ \alpha^2\tilde{P}_{2n}(\alpha z)\left[\tilde{P}_1(\alpha z)U_{2n+1}(\alpha z) - (\alpha z)^2U_{2n}(\alpha z)\right]$$

$$+\,\alpha\tilde{P}_{2n}(\alpha^2 z)\left[\tilde{P}_1(\alpha^2 z)U_{2n+1}(\alpha^2 z)-(\alpha^2 z)^2 U_{2n}(\alpha^2 z)\right]=0. \quad (3.4.15)$$

With the definition

$$W_{2n}(z):=\tilde{P}_1(z)U_{2n+1}(z)-z^2U_{2n}(z), \quad (3.4.16)$$

(3.4.15) becomes

$$\tilde{P}_{2n}(z)W_{2n}(z)+\alpha^2\tilde{P}_{2n}(\alpha z)W_{2n}(\alpha z)+\alpha\tilde{P}_{2n}(\alpha^2 z)W_{2n}(\alpha^2 z)=0. \quad (3.4.17)$$

Observe, from (3.3.2) and (3.3.4), that $W_{2n}\in\pi_{2n+2}$. Next, we set $z=1$ in (3.4.17) to obtain, by using also the fact that (3.3.2), (3.1.6) and (3.1.7) yield $\tilde{P}_{2n}(1)=1$ and $\tilde{P}_{2n}(\alpha)=\tilde{P}_{2n}(\alpha^2)=0$,

$$W_{2n}(1)=0,$$

from which we deduce that

$$W_{2n}(z)=(1-z)W_{2n-1}(z),$$

for some polynomial $W_{2n-1}\in\pi_{2n+1}$, so that (3.4.17) becomes

$$\tilde{P}_{2n}(z)(1-z)W_{2n-1}(z)+\alpha^2\tilde{P}_{2n}(\alpha z)(1-\alpha z)W_{2n-1}(\alpha z)$$
$$+\,\alpha\tilde{P}_{2n}(\alpha^2 z)(1-\alpha^2 z)W_{2n-1}(\alpha^2 z)=0. \quad (3.4.18)$$

We observe, by using (3.3.2), that

$$\begin{aligned}\tilde{P}_{2n}(z)(1-z)&=\tilde{P}_{2n-1}(z)\tilde{P}_1(z)(1-z)=\tfrac{1}{3}\tilde{P}_{2n-1}(z)(1-z^3);\\ \tilde{P}_{2n}(\alpha z)(1-\alpha z)&=\tilde{P}_{2n-1}(\alpha z)\tilde{P}_1(\alpha z)(1-\alpha z)=\tfrac{1}{3}\tilde{P}_{2n-1}(\alpha z)(1-z^3);\\ \tilde{P}_{2n}(\alpha^2 z)(1-\alpha^2 z)&=\tilde{P}_{2n-1}(\alpha^2 z)\tilde{P}_1(\alpha^2 z)(1-\alpha^2 z)=\tfrac{1}{3}\tilde{P}_{2n-1}(\alpha^2 z)(1-z^3),\end{aligned}$$

so that it follows from (3.4.18) that

$$\tilde{P}_{2n-1}(z)W_{2n-1}(z)+\alpha^2\tilde{P}_{2n-1}(\alpha z)W_{2n-1}(\alpha z)+\alpha\tilde{P}_{2n-1}(\alpha^2 z)W_{2n-1}(\alpha^2 z)=0. \quad (3.4.19)$$

Repeated applications of this procedure eventually yields a polynomial $W_0\in\pi_2$ such that

$$W_0(z)+\alpha^2W_0(\alpha z)+\alpha W_0(\alpha^2 z)=0, \quad (3.4.20)$$

and with

$$W_{2n}(z)=(1-z)^{2n}W_0(z). \quad (3.4.21)$$

Next, we note, from the symmetry properties of the polynomials $\tilde{P}_1$, U_{2n+1} and U_{2n}, as given, respectively, in (3.3.3) and (3.3.5), that, for $z\in\mathbb{C}\setminus\{0\}$,

$$\tilde{P}_1(z^{-1})=z^{-2}\tilde{P}_1(z);\quad U_{2n+1}(z^{-1})=z^{-2n}U_{2n+1}(z);\quad U_{2n}(z^{-1})=z^{-2n+2}U_{2n}(z),$$

and thus, from (3.4.16),

$$\begin{aligned} W_{2n}(z^{-1}) &= \tilde{P}_1(z^{-1})U_{2n+1}(z^{-1}) - z^{-2}U_{2n}(z^{-1}) \\ &= z^{-2}\tilde{P}_1(z)z^{-2n}U_{2n+1}(z) - z^{-2}z^{-2n+2}U_{2n}(z) \\ &= z^{-2n-2}\left[\tilde{P}_1(z)U_{2n+1}(z) - z^2U_{2n}(z)\right] = z^{-2n-2}W_{2n}(z). \end{aligned} \tag{3.4.22}$$

By combining (3.4.22) and (3.4.21), we obtain

$$\left(1 - z^{-1}\right)^{2n} W_0(z^{-1}) = z^{-2n-2}(1-z)^{2n}W_0(z), \quad z \in \mathbb{C} \setminus \{0\},$$

that is,

$$W_0(z^{-1}) = z^{-2}W_0(z), \quad z \in \mathbb{C} \setminus \{0\}. \tag{3.4.23}$$

But $W_0 \in \pi_2$, so that

$$W_0(z) = az^2 + bz + c, \tag{3.4.24}$$

for some $a, b, c \in \mathbb{R}$. By substituting (3.4.24) into (3.4.20), and using (3.1.6) and (3.1.7), we deduce that $b = 0$, so that

$$W_0(z) = az^2 + c, \tag{3.4.25}$$

for $a, c \in \mathbb{R}$, which, together with (3.4.23), yields

$$\tfrac{a}{z^2} + c = a + \tfrac{c}{z^2}, \quad z \in \mathbb{C} \setminus \{0\}.$$

This implies $c = a$, so that, from (3.4.25),

$$W_0(z) = K_2(1 + z^2),$$

for some $K_2 \in \mathbb{R}$, and thus, using also (3.4.21),

$$W_{2n}(z) = K_2(1-z)^{2n}(1+z^2). \tag{3.4.26}$$

It follows from (3.4.16) and (3.4.26) that

$$\tilde{P}_1(z)U_{2n+1}(z) - z^2U_{2n}(z) = K_2(1-z)^{2n}(1+z^2), \tag{3.4.27}$$

in which we now set $z = \alpha$ to obtain, by using also (3.3.2), (3.1.7) and (3.1.6),

$$K_2 = -\alpha^2U_{2n}(\alpha)\left[(1-\alpha)^2\right]^{-n}(1+\alpha^2)^{-1} = \frac{(-1)^n}{3^n}\alpha^{1-n}U_{2n}(\alpha). \tag{3.4.28}$$

By combining (3.4.27) and (3.4.28), and using also the definition (3.3.2), we obtain, for $z \in \mathbb{C} \setminus \{\alpha, \alpha^2\}$, the recursion formulation

$$U_{2n+1}(z) = \frac{3z^2U_{2n}(z) + \frac{(-1)^n}{3^{n-1}}\alpha^{1-n}U_{2n}(\alpha)(1-z)^{2n}(1+z^2)}{1+z+z^2}, \quad n \in \mathbb{N}. \tag{3.4.29}$$

We have therefore established the following result. An analogous result for binary subdivision is given in Theorem 7.2.2, p 280 in [1]. We note that the derivation and result here involve non-trivial extensions of the proof in [1].

Theorem 3.4.1 *The polynomial sequence $\{U_m : m \in \mathbb{N}\}$, as obtained from (3.2.29), (3.2.30), (3.2.26), satisfies, for $n = 1, 2, \ldots,$ and $z \in \mathbb{C} \setminus \{\alpha, \alpha^2\}$, the recursion formulation*

$$U_1(z) = 1; \tag{3.4.30}$$

$$U_{2n}(z) = \frac{3zU_{2n-1}(z) + \frac{(-1)^{n+1}}{3^{n-1}}\alpha^{1-n}U_{2n-1}(\alpha)(1-z)^{2n}}{1+z+z^2}; \tag{3.4.31}$$

$$U_{2n+1}(z) = \frac{3z^2U_{2n}(z) + \frac{(-1)^{n}}{3^{n-1}}\alpha^{1-n}U_{2n}(\alpha)(1-z)^{2n}(1+z^2)}{1+z+z^2}. \tag{3.4.32}$$

We note that the recursive formulation in Theorem 3.4.1 provides a more efficient method to compute the polynomials $\{U_m : m \in \mathbb{N}\}$ than the explicit formulation (3.2.29), (3.2.30), (3.2.26). By using the recursive formulation, the polynomials U_m for $m = 1, \ldots, 10$, are computed and compiled in Table 3.1.

We observe that, for $m = 1, \ldots, 10$, the degree of the polynomial U_m is precisely $m-1$ if m is odd and $m-2$ if m is even. Also, for $m = 1, \ldots, 10$,

$$U_m(\alpha) \neq 0; \quad U_m(\alpha^2) \neq 0,$$

implying

$$R(\alpha) \neq 0; \quad R(\alpha^2) \neq 0,$$

from (3.2.4), which can be shown to be equivalent to the sum-rule condition of order precisely $m \in \mathbb{N}$ on the refinement sequence $\{p_{m,j}\}$. It is our conjecture that these two observations are true for any $m \in \mathbb{N}$, as will be investigated in further research.

m	U_m
1	1
2	1
3	$-1+3z-z^2$
4	$\frac{1}{3}(-4+11z-4z^2)$
5	$\frac{1}{9}(15-75z+129z^2-75z^3+15z^4)$
6	$\frac{1}{3}(7-34z+57z^2-34z^3+7z^4)$
7	$\frac{1}{9}(-28+196z-553z^2+779z^3-553z^4+196z^5-28z^6)$
8	$\frac{1}{27}(-120+828z-2304z^2+3219z^3-2304z^4+828z^5-120z^6)$
9	$\frac{1}{27}(165-1485z+5745z^2-12336z^3+15849z^4-12336z^5+5745z^6-1485z^7+165z^8)$
10	$\frac{1}{81}(715-6380z+24475z^2-52190z^3+66841z^4-52190z^5+24475z^6-6380z^7+715z^8)$

Table 3.1: The polynomial U_m for $m = 1, \ldots, 10$.

Chapter 4

Convergence analysis

In the previous chapter, we derived a necessary condition on a refinement sequence $\{p_j\}$, such that its corresponding refinable function ϕ is interpolatory (see Corollary 3.2.3). However, we still need to show that this necessary condition is also sufficient to ensure that there exists an interpolatory refinable function ϕ, that is, we need to show that if $\{p_j\} = \{p_{m,j}\}$ is given by (3.2.38) in Corollary 3.2.3, then there exists a corresponding interpolatory refinable function ϕ. We will do this by deriving a convergence criterion on the refinement sequence $\{p_{m,j}\}$ of Corollary 3.2.3, which, when satisfied, will ensure that the corresponding interpolatory subdivision operator provides a convergent subdivision scheme with limit function $\phi_m := \phi_{\boldsymbol{p}_m}$. By Corollary 2.4.15, we know that the limit function ϕ_m is a refinable function with refinement sequence $\{p_{m,j}\}$; in this chapter, we will show that ϕ_m is also interpolatory.

4.1 Cascade operator

Our main result in this section will be to derive a sufficient condition on a refinement sequence $\{p_j\}$ such that the corresponding subdivision operator $\mathcal{S}_{\boldsymbol{p}}$ provides a convergent subdivision scheme. To this end, we define the notion of the cascade operator, as follows.

Definition 4.1.1 *Let $\boldsymbol{p} = \{p_j\} \in l_0$ and $f \in C(\mathbb{R})$. We define the cascade operator $\mathcal{C}_{\boldsymbol{p}}$ corresponding to $\boldsymbol{p}$ by*

$$\left(\mathcal{C}_{\boldsymbol{p}} f\right)(x) := \sum_j p_j f(3x - j), \quad x \in \mathbb{R}. \tag{4.1.1}$$

Observe from (4.1.1) and (2.1.3) that a refinable function can be interpreted as a fixed point of the cascade operator $\mathcal{C}_{\boldsymbol{p}}$. We will rely on the following properties of the cascade operator, the proof of which is a straightforward adaptation of the proof of the analogous result in binary subdivision (see Lemma 6.1.1, p 206 in [1]).

Lemma 4.1.2 *Let $\boldsymbol{p} = \{p_j\} \in l_0$, with $\operatorname{supp}\{p_j\} = [\mu, \nu]\big|_{\mathbb{Z}}$. Then the following hold:*

(i) If $f \in C_0$, with $\operatorname{supp}^c f = [\sigma, \tau]$ for integers $\sigma < \tau$, then $\mathcal{C}_{\boldsymbol{p}} f \in C_0$, with

$$\operatorname{supp}^c(\mathcal{C}_{\boldsymbol{p}} f) = \left[\tfrac{1}{3}(\sigma + \mu), \tfrac{1}{3}(\tau + \nu)\right]; \tag{4.1.2}$$

(ii) If $\{p_j\}$ satisfies the sum-rule condition (2.2.8), and $f \in C_0$ is such that f provides a partition of unity, that is,

$$\sum_j f(x - j) = 1, \quad x \in \mathbb{R}, \tag{4.1.3}$$

then $\mathcal{C}_{\boldsymbol{p}} f$ also provides a partition of unity, that is,

$$\sum_j (\mathcal{C}_{\boldsymbol{p}} f)(x - j) = 1, \quad x \in \mathbb{R}. \tag{4.1.4}$$

Proof.

(i) For $f \in C_0$ with $\operatorname{supp}^c f = [\sigma, \tau]$, the result follows immediately from (4.1.1) and the fact that $\operatorname{supp}\{p_j\} = [\mu, \nu]\big|_{\mathbb{Z}}$.

(ii) Let $\{p_j\} \in l_0$ such that the sum-rule condition (2.2.8), or, equivalently,

$$\sum_j p_{k-3j} = 1, \tag{4.1.5}$$

is satisfied. Also, let $f \in C_0$ such that (4.1.3) holds. It follows from (4.1.1), (4.1.5) and (4.1.3), for any $x \in \mathbb{R}$, that

$$\begin{aligned}
\sum_j (\mathcal{C}_{\boldsymbol{p}} f)(x - j) &= \sum_j \left(\sum_k p_k f(3x - 3j - k) \right) \\
&= \sum_j \left(\sum_k p_{k-3j} f(3x - k) \right) \\
&= \sum_k \left(\sum_j p_{k-3j} \right) f(3x - k) \\
&= \sum_k f(3x - k) = 1,
\end{aligned}$$

and thereby completing our proof. ■

In this chapter, we shall say that a sequence $\{p_j\} \in l_0$ is centered if the integers μ and ν in $\text{supp}\{p_j\} = [\mu, \nu]\big|_{\mathbb{Z}}$ satisfy

$$\mu \leq -2; \quad \nu \geq 2. \tag{4.1.6}$$

We proceed to define the cascade algorithm, based on the cascade operator $\mathcal{C}_{\boldsymbol{p}}$.

Definition 4.1.3 *For a given centered sequence $\boldsymbol{p} = \{p_j\} \in l_0$, let the function sequence $\{h_r : r = 0, 1, \ldots\}$ be generated recursively by*

$$h_0 := h; \quad h_r := \mathcal{C}_{\boldsymbol{p}} h_{r-1} = \mathcal{C}_{\boldsymbol{p}}^r h_0, \quad r = 1, 2, \ldots, \tag{4.1.7}$$

with h denoting the hat function, as given in (2.1.6). The recursive scheme (4.1.7) is then called the cascade algorithm corresponding to $\{p_j\}$.

By using Lemma 4.1.2, we can now prove the following properties of functions generated by the cascade algorithm. The proof follows the same pattern as the proofs of the analogous results in binary subdivision (see Theorem 6.1.1, p 208 and Theorem 8.1.2, p 297 in [1]).

Theorem 4.1.4 *Let $\boldsymbol{p} = \{p_j\} \in l_0$ be a centered sequence, with* $\text{supp}\{p_j\} = [\mu, \nu]\big|_{\mathbb{Z}}$, *and let $\{h_r : r = 1, 2, \ldots\}$ denote the functions generated by the cascade algorithm, as defined in Definition 4.1.3. Then the following hold:*

(i) For $r = 0, 1, \ldots,$

$$h_r(x) = 0, \quad x \notin (\tfrac{\mu}{2}, \tfrac{\nu}{2}), \tag{4.1.8}$$

with, more precisely,

$$\text{supp}^c h_r = \left[\tfrac{\mu}{2} - \tfrac{(\mu/2)+1}{3^r}, \tfrac{\nu}{2} - \tfrac{(\nu/2)-1}{3^r}\right]; \tag{4.1.9}$$

(ii) If $\{p_j\}$ satisfies the sum-rule condition (2.2.8), then, for $r = 0, 1, \ldots,$ h_r provides a partition of unity, that is,

$$\sum_j h_r(x - j) = 1, \quad x \in \mathbb{R}; \tag{4.1.10}$$

(iii) For $r = 1, 2, \ldots,$

$$h_r(x) = \sum_j p_j^{[r]} h(3^r x - j), \quad x \in \mathbb{R}, \tag{4.1.11}$$

and

$$h_r(\tfrac{j}{3^r}) = p_j^{[r]}, \tag{4.1.12}$$

with $\left\{p_j^{[r]} : j \in \mathbb{Z}\right\}$ defined by (2.2.18), and h denoting the hat function in (2.1.6);

(iv) If $\{p_j\}$ satisfies the interpolatory condition (3.1.2), then, for $r = 0, 1, \ldots,$

$$h_r(j) = \delta_j, \quad j \in \mathbb{Z}, \tag{4.1.13}$$

with $\boldsymbol{\delta} = \{\delta_j\}$ denoting the Kronecker delta sequence, as in (2.2.13).

Proof.

(i) Our proof is by induction on r. If $r = 0$, we have

$$\text{supp}^c h_0 = \text{supp}^c h = [-1, 1] = \left[\tfrac{\mu}{2} - \tfrac{(\mu/2)+1}{3^0}, \tfrac{\nu}{2} - \tfrac{(\nu/2)-1}{3^0}\right],$$

and

$$h_0(x) = h(x) = 0, \quad x \notin (\tfrac{\mu}{2}, \tfrac{\nu}{2}),$$

from (4.1.7) and (2.1.6) in Example 2.1.4, by using also (4.1.6), so that the result holds for $r = 0$. Next, suppose the result holds for some $r \in \mathbb{N}$. It follows from (4.1.7) and (4.1.2) in Lemma 4.1.2 (i), together with the inductive assumption, that

$$\begin{aligned}\text{supp}^c h_{r+1} = \text{supp}^c\left(\mathcal{C}_{\boldsymbol{p}} h_r\right) &= \left[\tfrac{1}{3}\left(\tfrac{\mu}{2} - \tfrac{(\mu/2)+1}{3^r} + \mu\right), \tfrac{1}{3}\left(\tfrac{\nu}{2} - \tfrac{(\nu/2)-1}{3^r} + \nu\right)\right] \\ &= \left[\tfrac{\mu}{2} - \tfrac{(\mu/2)+1}{3^{r+1}}, \tfrac{\nu}{2} - \tfrac{(\nu/2)-1}{3^{r+1}}\right],\end{aligned}$$

so that (4.1.9) holds. Moreover, for μ and ν satisfying (4.1.6), we have

$$\text{supp}^c h_r = \left[\tfrac{\mu}{2} - \tfrac{(\mu/2)+1}{3^r}, \tfrac{\nu}{2} - \tfrac{(\nu/2)-1}{3^r}\right] \subset \left[\tfrac{\mu}{2}, \tfrac{\nu}{2}\right], \quad r = 0, 1, \ldots,$$

and thus

$$h_r(x) = 0, \quad x \notin \left[\tfrac{\mu}{2}, \tfrac{\nu}{2}\right].$$

By the continuity of h_r at $\frac{\mu}{2}$ and $\frac{\nu}{2}$ for each $r = 0, 1, \ldots,$ from (i) in Lemma 4.1.2, we have

$$h_r\left(\tfrac{\mu}{2}\right) = 0; \quad h_r\left(\tfrac{\nu}{2}\right) = 0,$$

so that

$$h_r(x) = 0, \quad x \notin \left(\tfrac{\mu}{2}, \tfrac{\nu}{2}\right),$$

and thereby completing our proof of (i).

(ii) We have already shown in Example 2.1.4 that the hat function h provides a partition of unity. Now assume that h_r provides a partition of unity for some $r \in \mathbb{N}$. It then follows from (4.1.7), the induction hypothesis and Lemma 4.1.2 (ii) that

$$\sum_j h_{r+1}(x - j) = \sum_j \mathcal{C}_{\boldsymbol{p}} h_r(x - j) = 1,$$

that is, h_{r+1} provides a partition of unity, and thereby completing our inductive proof.

(iii) We first prove (4.1.11) by induction on r. If $r = 1$, we have, from (4.1.7), (4.1.1) and (2.3.15) in Theorem 2.3.2,

$$h_1(x) = (\mathcal{C}_{\boldsymbol{p}} h)(x) = \sum_j p_j h(3x - j) = \sum_j p_j^{[1]} h(3x - j),$$

that is, (4.1.11) holds for $r = 1$. Now assume that (4.1.11) holds for some $r \geq 2$. By using (2.3.15) and the inductive assumption, and finally (4.1.1) and (4.1.7), we obtain

$$\begin{aligned}
\sum_j p_j^{[r+1]} h(3^{r+1}x - j) &= \sum_j \left[\sum_k p_k p_{j-3^r k}^{[r]} \right] h(3^{r+1}x - j) \\
&= \sum_k p_k \left[\sum_j p_{j-3^r k}^{[r]} h(3^{r+1}x - j) \right] \\
&= \sum_k p_k \left[\sum_j p_j^{[r]} h(3^{r+1}x - j - 3^r k) \right] \\
&= \sum_k p_k \left[\sum_j p_j^{[r]} h(3^r(3x - k) - j) \right] \\
&= \sum_k p_k h_r(3x - k) \\
&= (\mathcal{C}_{\boldsymbol{p}} h_r)(x) = h_{r+1}(x).
\end{aligned}$$

To prove (4.1.12), we use (4.1.11), together with (2.1.6), to obtain, for $r = 1, 2, \ldots,$

$$\begin{aligned}
h_r\left(\tfrac{k}{3^r}\right) = \sum_j p_j^{[r]} h\left(3^r\left(\tfrac{k}{3^r}\right) - j\right) &= \sum_j p_j^{[r]} h(k - j) \\
&= p_k^{[r]} h(0) = p_k^{[r]}(1) = p_k^{[r]}.
\end{aligned}$$

(iv) Our proof is by induction on r. If $r = 0$, it follows from (4.1.7), together with (2.1.6), that

$$h_0(j) = h(j) = \delta_j, \quad j \in \mathbb{Z},$$

so that (4.1.13) holds for $r = 0$. Assume the result holds for some $r \geq 1$. By applying (4.1.7) and (4.1.1), together with the inductive assumption, (2.2.13) and (3.1.2), we obtain, for all $j \in \mathbb{Z}$,

$$\begin{aligned}
h_{r+1}(j) = (\mathcal{C}_{\boldsymbol{p}} h_r)(j) &= \sum_k p_k h_r(3j - k) \\
&= \sum_k p_{3j-k} h_r(k)
\end{aligned}$$

$$= \sum_k p_{3j-k}\delta_k$$
$$= p_{3j} = \delta_j,$$

and thereby completing our proof. ■

Convergence of the cascade algorithm is defined as follows.

Definition 4.1.5 *Let $\boldsymbol{p} = \{p_j\} \in l_0$ be a centered sequence. The cascade algorithm based on $\boldsymbol{p}$, as given in (4.1.7), is said to be convergent if there exists a function $h_{\boldsymbol{p}} \in C(\mathbb{R})$ such that*

$$||h_{\boldsymbol{p}} - h_r||_\infty = \sup_x |h_{\boldsymbol{p}}(x) - h_r(x)| \to 0, \quad r \to \infty. \tag{4.1.14}$$

We call $h_{\boldsymbol{p}}$ the limit function of the cascade algorithm.

By using Theorem 4.1.4, we now prove the following properties of the limit function $h_{\boldsymbol{p}}$ in Definition 4.1.5. The proof follows the same pattern as the proofs of the analogous results in binary subdivision (see Lemma 6.1.2, p 210 and Theorem 8.1.2, p 297 in [1]).

Lemma 4.1.6 *Let $\boldsymbol{p} = \{p_j\} \in l_0$ be a centered sequence, with* $\text{supp}\{p_j\} = [\mu, \nu]\big|_{\mathbb{Z}}$. *Suppose the cascade algorithm based on $\boldsymbol{p}$ is convergent with limit function $h_{\boldsymbol{p}}$. Then the following hold:*

(i) $h_{\boldsymbol{p}} \in C_0$, with

$$h_{\boldsymbol{p}}(x) = 0, \quad x \notin \left(\tfrac{\mu}{2}, \tfrac{\nu}{2}\right); \tag{4.1.15}$$

(ii) If $\{p_j\}$ satisfies the sum-rule condition (2.2.8), then $h_{\boldsymbol{p}}$ provides a partition of unity, that is,

$$\sum_j h_{\boldsymbol{p}}(x-j) = 1, \quad x \in \mathbb{R}; \tag{4.1.16}$$

(iii) If $\{p_j\}$ satisfies the interpolatory condition (3.1.2), then

$$h_{\boldsymbol{p}}(j) = \delta_j, \quad j \in \mathbb{Z}, \tag{4.1.17}$$

with $\boldsymbol{\delta} = \{\delta_j\}$ denoting the Kronecker delta sequence, as in (2.2.13).

Proof.

(i) Let $x \in \mathbb{R}$, with $x \notin \left(\frac{\mu}{2}, \frac{\nu}{2}\right)$. By using (4.1.8) in Theorem 4.1.4 (i) and (4.1.14), we obtain

$$0 \leq |h_{\boldsymbol{p}}(x)| = |h_{\boldsymbol{p}}(x) - h_r(x)| \leq ||h_{\boldsymbol{p}} - h_r||_\infty \to 0, \quad r \to \infty,$$

from which (4.1.15) follows.

(ii) Let $x \in \mathbb{R}$, and denote by k the (unique) integer such that

$$k \leq x < k+1. \tag{4.1.18}$$

It follows from (4.1.10) in Theorem 4.1.4 (ii), (4.1.8), (4.1.15), (4.1.18) and (4.1.14), that

$$\begin{aligned} 0 \leq \Big|\sum_j h_{\boldsymbol{p}}(x-j) - 1\Big| &= \Big|\sum_j h_{\boldsymbol{p}}(x-j) - \sum_j h_r(x-j)\Big| \\ &= \left|\sum_{j=k+1-\lceil \nu/2 \rceil}^{k-\lfloor \mu/2 \rfloor} \left(h_{\boldsymbol{p}}(x-j) - h_r(x-j)\right)\right| \\ &\leq \sum_{j=k+1-\lceil \nu/2 \rceil}^{k-\lfloor \mu/2 \rfloor} |h_{\boldsymbol{p}}(x-j) - h_r(x-j)| \\ &\leq ||h_{\boldsymbol{p}} - h_r||_\infty \left(\lceil \tfrac{\nu}{2} \rceil - \lfloor \tfrac{\mu}{2} \rfloor\right) \to 0, \quad r \to \infty, \end{aligned}$$

so that (4.1.16) follows.

(iii) By applying (4.1.13) in Theorem 4.1.4 (iv), together with (4.1.14), we obtain, for all $j \in \mathbb{Z}$ and $r = 0, 1, \ldots,$

$$0 \leq |h_{\boldsymbol{p}}(j) - \delta_j| = |h_{\boldsymbol{p}}(j) - h_r(j)| \leq ||h_{\boldsymbol{p}} - h_r||_\infty \to 0, \quad r \to \infty,$$

so that (4.1.17) follows. ∎

The main result of this section is as follows. The proof is similar to the proof of the analogous result in binary subdivision (see Theorem 6.1.2 on p 210 in [1]).

Theorem 4.1.7 *Let $\boldsymbol{p} = \{p_j\} \in l_0$ be a centered sequence such that the sum-rule condition (2.2.8) is satisfied. If the cascade algorithm based on $\boldsymbol{p}$, as given in (4.1.7), is convergent with limit function $h_{\boldsymbol{p}}$, then the subdivision operator $\mathcal{S}_{\boldsymbol{p}}$, as defined by (2.2.14), provides a convergent subdivision scheme, with limit function*

$$\phi_{\boldsymbol{p}} := h_{\boldsymbol{p}}, \tag{4.1.19}$$

and

$$\sup_j \left|\phi_{\boldsymbol{p}}\left(\tfrac{j}{3^r}\right) - p_j^{[r]}\right| \leq ||h_{\boldsymbol{p}} - h_r||_\infty, \tag{4.1.20}$$

for $r = 1, 2, \ldots,$ with $\left\{p_j^{[r]} : j \in \mathbb{Z}\right\}$ defined by (2.2.18). If, moreover, $\{p_j\}$ satisfies the interpolatory condition (3.1.2), then the limit function $\phi_{\boldsymbol{p}}$ is interpolatory, that is,

$$\phi_{\boldsymbol{p}}(j) = \delta_j, \quad j \in \mathbb{Z}, \tag{4.1.21}$$

with $\boldsymbol{\delta} = \{\delta_j\}$ denoting the Kronecker delta sequence, as in (2.2.13).

Proof.
For $r = 1, 2, \ldots$, we have

$$\sup_j \left| h_{\boldsymbol{p}}\left(\tfrac{j}{3^r}\right) - h_r\left(\tfrac{j}{3^r}\right)\right| \leq \sup_x |h_{\boldsymbol{p}}(x) - h_r(x)|,$$

and thus, by recalling (4.1.12) in Theorem 4.1.4 (iii),

$$\sup_j \left| h_{\boldsymbol{p}}\left(\tfrac{j}{3^r}\right) - p_j^{[r]}\right| \leq \sup_x |h_{\boldsymbol{p}}(x) - h_r(x)|, \tag{4.1.22}$$

with $\left\{p_j^{[r]} : j \in \mathbb{Z}\right\}$ defined by (2.2.18). Define

$$\phi_{\boldsymbol{p}} := h_{\boldsymbol{p}},$$

and note that $h_{\boldsymbol{p}}$ is non-trivial by virtue of (4.1.16) in Lemma 4.1.6 (ii). Then (4.1.22) becomes

$$\sup_j \left| \phi_{\boldsymbol{p}}\left(\tfrac{j}{3^r}\right) - p_j^{[r]}\right| \leq \sup_x |h_{\boldsymbol{p}}(x) - h_r(x)|,$$

from which it follows, by using also (4.1.14), that

$$\sup_j \left| \phi_{\boldsymbol{p}}\left(\tfrac{j}{3^r}\right) - p_j^{[r]}\right| \leq ||h_{\boldsymbol{p}} - h_r||_\infty \to 0, \quad r \to \infty,$$

that is, $\mathcal{S}_{\boldsymbol{p}}$ provides a convergent subdivision scheme with limit function $\phi_{\boldsymbol{p}}$.

Moreover, if $\{p_j\}$ satisfies the interpolatory condition (3.1.2), then (4.1.21) follows immediately from (4.1.17) in Lemma 4.1.6 (iii), together with (4.1.19), and thereby completing our proof. ■

It follows from Theorem 4.1.7 that, in order to prove the existence of an interpolatory refinable function ϕ with refinement sequence $\{p_{m,j}\}$, it will suffice to show that the cascade algorithm based on $\{p_{m,j}\}$ is convergent with limit function $h_m := h_{\boldsymbol{p}_m}$, and set $\phi := \phi_m = h_m$. In the next section, we will obtain a sufficient condition to ensure that the cascade algorithm based on $\{p_{m,j}\}$ is indeed convergent.

4.2 Contractivity condition

It was shown in [7], Theorem 2.26, p 32, that if $\{p_j\}$ is a sequence in l_0 with only positive terms, then the cascade algorithm based on $\{p_j\}$ is convergent. However, for $\{p_{m,j}\}$ given by (3.2.38) in Corollary 3.2.3, the terms of $\{p_{m,j}\}$ are not necessarily all positive. In this section, we will focus our attention on interpolatory subdivision schemes, and derive a certain contractivity condition which, when satisfied, will ensure the convergence of the cascade algorithm based on the interpolatory refinement sequence $\{p_{m,j}\}$ of Corollary 3.2.3.

We start by defining the notion of a bounded linear operator (see e.g. [3], pp 87-88).

Definition 4.2.1 *For a normed linear space $(X, ||\cdot||)$, let $T : X \to X$ be a linear operator. If*

$$||T|| := \sup\left\{\frac{||Tf||}{||f||} : f \in X,\ f \neq 0\right\} < \infty, \tag{4.2.1}$$

we say that T is a bounded linear operator.

Note from (4.2.1) that, if $T : X \to X$ is a bounded linear operator, then

$$||Tf|| \leq ||T||\,||f||, \quad f \in X. \tag{4.2.2}$$

We proceed to obtain an explicit formulation of the uniform norm of the subdivision operator defined by (2.2.14). The proof is similar to the proof of the analogous result in binary subdivision (see Theorem 2.1, p 28 in [17]).

Theorem 4.2.2 *Let $\boldsymbol{p} = \{p_j\} \in l_0$, and let $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in l(\mathbb{Z})$ be a sequence of control points. The subdivision operator $\mathcal{S}_{\boldsymbol{p}}$ is a bounded linear operator from l^∞ into itself, with*

$$||\mathcal{S}_{\boldsymbol{p}}||_\infty = \rho_{\boldsymbol{p}} := \max\left\{\sum_j |p_{3j}|, \sum_j |p_{3j+1}|, \sum_j |p_{3j+2}|\right\}, \tag{4.2.3}$$

and

$$||\mathcal{S}_{\boldsymbol{p}}\boldsymbol{c}||_\infty \leq \rho_{\boldsymbol{p}}||\boldsymbol{c}||_\infty. \tag{4.2.4}$$

Proof.
Let $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in l^\infty$. From (2.2.14), we obtain, for all $j \in \mathbb{Z}$,

$$\left.\begin{aligned}
|\,(\mathcal{S}_{\boldsymbol{p}}\boldsymbol{c})_{3j}\,| &= \Big|\sum_k p_{3j-3k}\boldsymbol{c}_k\Big| = \Big|\sum_k p_{3k}\boldsymbol{c}_{j-k}\Big| \leq ||\boldsymbol{c}||_\infty \sum_k |p_{3k}|;\\
|\,(\mathcal{S}_{\boldsymbol{p}}\boldsymbol{c})_{3j+1}\,| &= \Big|\sum_k p_{3j+1-3k}\boldsymbol{c}_k\Big| = \Big|\sum_k p_{3k+1}\boldsymbol{c}_{j-k}\Big| \leq ||\boldsymbol{c}||_\infty \sum_k |p_{3k+1}|;\\
|\,(\mathcal{S}_{\boldsymbol{p}}\boldsymbol{c})_{3j+2}\,| &= \Big|\sum_k p_{3j+2-3k}\boldsymbol{c}_k\Big| = \Big|\sum_k p_{3k+2}\boldsymbol{c}_{j-k}\Big| \leq ||\boldsymbol{c}||_\infty \sum_k |p_{3k+2}|,
\end{aligned}\right\}$$

and thus

$$|\,(\mathcal{S}_{\boldsymbol{p}}\boldsymbol{c})_j\,| \leq ||\boldsymbol{c}||_\infty\rho_{\boldsymbol{p}}, \quad j \in \mathbb{Z}, \tag{4.2.5}$$

with

$$\rho_{\boldsymbol{p}} := \max\left\{\sum_j |p_{3j}|, \sum_j |p_{3j+1}|, \sum_j |p_{3j+2}|\right\}, \tag{4.2.6}$$

from which (4.2.4) then follows.

We see from (4.2.5) that $\left\{(\mathcal{S}_{\boldsymbol{p}}\boldsymbol{c})_j : j \in \mathbb{Z}\right\}$ is a bounded sequence, that is, $\mathcal{S}_{\boldsymbol{p}}$ maps l^∞ into itself. Also, it follows from the definition (2.2.14) that $\mathcal{S}_{\boldsymbol{p}}$ is a linear operator, and we observe from (4.2.4) that

$$\frac{||\mathcal{S}_{\boldsymbol{p}}\boldsymbol{c}||_\infty}{||\boldsymbol{c}||_\infty} \leq \rho_{\boldsymbol{p}}, \quad \boldsymbol{c} \in l^\infty, \ \boldsymbol{c} \neq \boldsymbol{0},$$

that is, as in (4.2.1), $\mathcal{S}_{\boldsymbol{p}}$ is a bounded operator, with

$$||\mathcal{S}_{\boldsymbol{p}}||_\infty \leq \rho_{\boldsymbol{p}}. \tag{4.2.7}$$

We proceed to prove the inequality

$$||\mathcal{S}_{\boldsymbol{p}}||_\infty \geq \rho_{\boldsymbol{p}},$$

from which, together with (4.2.7), the desired result (4.2.3) will then follow, and thereby completing our proof. To this end, let $j \in \mathbb{Z}$ be fixed, and define the sequence $\boldsymbol{c}_j = \{\boldsymbol{c}_{j,k} : k \in \mathbb{Z}\} \in l^\infty$, with $\boldsymbol{c}_{j,k} \in \mathbb{R}^s$, $k \in \mathbb{Z}$, for some integer $s \in \mathbb{N}$, by

$$\boldsymbol{c}_j = \{\boldsymbol{c}_{j,k} : k \in \mathbb{Z}\} := \{(\tilde{c}_{j,k}, 0, 0, \ldots, 0) : k \in \mathbb{Z}\}, \tag{4.2.8}$$

where

$$\tilde{c}_{j,k} := \begin{cases} 1, & \text{if } p_{j-3k} \geq 0; \\ -1, & \text{if } p_{j-3k} < 0. \end{cases} \tag{4.2.9}$$

Note that then

$$||\boldsymbol{c}_j||_\infty = \sup_k |\boldsymbol{c}_{j,k}| = |\tilde{c}_{j,k}| = 1. \tag{4.2.10}$$

By applying (4.2.9), (4.2.8), (2.2.14), (4.2.1) and (4.2.10), we obtain

$$\begin{aligned}
\sum_k |p_{j-3k}| = \sum_k p_{j-3k}\tilde{c}_{j,k} &= \Big|\sum_k p_{j-3k}\tilde{c}_{j,k}\Big| \\
&= \Big|\sum_k p_{j-3k}\left(\tilde{c}_{j,k}, 0, \ldots, 0\right)\Big| \\
&= \Big|\sum_k p_{j-3k}\boldsymbol{c}_j\Big| \\
&= \left|\left(\mathcal{S}_{\boldsymbol{p}}\boldsymbol{c}_j\right)_j\right| \\
&\leq \sup_k \left|\left(\mathcal{S}_{\boldsymbol{p}}\boldsymbol{c}_j\right)_k\right| \\
&= ||\mathcal{S}_{\boldsymbol{p}}\boldsymbol{c}_j||_\infty
\end{aligned}$$

$$\leq ||\mathcal{S}_{\boldsymbol{p}}||_\infty ||\boldsymbol{c}_j||_\infty = ||\mathcal{S}_{\boldsymbol{p}}||_\infty,$$

and thus

$$\sup_j \left(\sum_k |p_{j-3k}| \right) \leq ||\mathcal{S}_{\boldsymbol{p}}||_\infty. \tag{4.2.11}$$

Now observe that, for $j \in \mathbb{Z}$,

$$\left.\begin{aligned} \sum_k |p_{3j-3k}| &= \sum_k |p_{3k}|; \\ \sum_k |p_{3j+1-3k}| &= \sum_k |p_{3k+1}|; \\ \sum_k |p_{3j+2-3k}| &= \sum_k |p_{3k+2}|. \end{aligned}\right\} \tag{4.2.12}$$

By applying (4.2.12) and (4.2.6) in (4.2.11), we obtain

$$\begin{aligned} \sup_j \left(\sum_k |p_{j-3k}| \right) &= \max \left\{ \sum_j |p_{3j}|, \sum_j |p_{3j+1}|, \sum_j |p_{3j+2}| \right\} \\ &= \rho_{\boldsymbol{p}} \leq ||\mathcal{S}_{\boldsymbol{p}}||_\infty, \end{aligned}$$

and thereby completing our proof. ■

We will rely on the following lemmas in our subsequent subdivision convergence analysis. The proofs of Lemmas 4.2.4 - 4.2.6 follow the same patterns as the proofs of the analogous results in binary subdivision (see Lemmas 3.3, 3.4 and 3.5, pp 43 - 46 in [17]).

Lemma 4.2.3 *For $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in l(\mathbb{Z})$ and any $k \in \mathbb{N}$, the k^{th} backward difference operator Δ^k, as defined by (2.4.1) and (2.4.2), satisfies the formulation*

$$\left(\Delta^k \boldsymbol{c}\right)_j = \sum_{l=0}^{k} (-1)^l \binom{k}{l} \boldsymbol{c}_{j-l}, \quad j \in \mathbb{Z}. \tag{4.2.13}$$

Proof.
Our proof is by induction on k. If $k = 1$, (4.2.13) follows immediately from the definition (2.4.1). Assume the result holds for some $k \geq 2$. By applying (2.4.2), (2.4.1) and the inductive assumption, we obtain, for all $j \in \mathbb{Z}$,

$$\begin{aligned} \left(\Delta^{k+1} \boldsymbol{c}\right)_j &= \left(\Delta \left(\Delta^k \boldsymbol{c}\right)\right)_j \\ &= \left(\Delta^k \boldsymbol{c}\right)_j - \left(\Delta^k \boldsymbol{c}\right)_{j-1} \\ &= \sum_{l=0}^{k} (-1)^l \binom{k}{l} \boldsymbol{c}_{j-l} - \sum_{l=0}^{k} (-1)^l \binom{k}{l} \boldsymbol{c}_{j-1-l} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{l=0}^{k}(-1)^l\binom{k}{l}\boldsymbol{c}_{j-l} - \sum_{l=1}^{k+1}(-1)^{l-1}\binom{k}{l-1}\boldsymbol{c}_{j-l} \\
&= \sum_{l=0}^{k+1}\left[(-1)^l\binom{k}{l} + (-1)^l\binom{k}{l-1}\right]\boldsymbol{c}_{j-l} = \sum_{l=0}^{k+1}(-1)^l\binom{k+1}{l}\boldsymbol{c}_{j-l},
\end{aligned}
$$

and thereby completing our inductive proof. ■

Lemma 4.2.4 *Let $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in l_0$, and define the Laurent polynomial*

$$
C(z) := \sum_j \boldsymbol{c}_j z^j, \quad z \in \mathbb{C} \setminus \{0\}. \tag{4.2.14}
$$

Then, for $l \in \mathbb{N}$,

$$
\sum_j \left(\Delta^l \boldsymbol{c}\right)_j z^j = (1-z)^l C(z), \quad z \in \mathbb{C} \setminus \{0\}. \tag{4.2.15}
$$

Proof.

By using (4.2.13) in Lemma 4.2.3, together with (4.2.14), we obtain, for $l \in \mathbb{N}$ and any $z \in \mathbb{C} \setminus \{0\}$,

$$
\begin{aligned}
\sum_j \left(\Delta^l \boldsymbol{c}\right)_j z^j &= \sum_j \left[\sum_{k=0}^{l}(-1)^k\binom{l}{k}\boldsymbol{c}_{j-k}\right] z^j \\
&= \sum_{k=0}^{l}(-1)^k\binom{l}{k}\left[\sum_j \boldsymbol{c}_{j-k} z^{j-k}\right] z^k \\
&= \sum_{k=0}^{l}(-1)^k\binom{l}{k}\left[\sum_j \boldsymbol{c}_j z^j\right] z^k \\
&= C(z)\left[\sum_{k=0}^{l}\binom{l}{k}(-z)^k\right] = C(z)\,(1-z)^l.
\end{aligned}
$$

■

Lemma 4.2.5 *Let $\boldsymbol{p} = \{p_j\} \in l_0$ and $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in l_0$, and let the Laurent polynomials P and C be defined by, respectively, (3.1.4) and (4.2.14). Then*

$$
\sum_j \left(\mathcal{S}_{\boldsymbol{p}} \boldsymbol{c}\right)_j z^j = 3P(z)C(z^3), \quad z \in \mathbb{C} \setminus \{0\}. \tag{4.2.16}
$$

Proof.
It follows from (2.2.14), together with the definitions (3.1.4) and (4.2.14), that

$$\begin{aligned}\sum_j (\mathcal{S}_{\boldsymbol{p}}\boldsymbol{c})_j z^j &= \sum_j \left[\sum_k p_{j-3k}\boldsymbol{c}_k\right] z^j \\ &= \sum_k \boldsymbol{c}_k \left[\sum_j p_{j-3k} z^{j-3k}\right] z^{3k} \\ &= \sum_k \boldsymbol{c}_k \left[\sum_j p_j z^j\right] z^{3k} \\ &= 3P(z)\sum_k \boldsymbol{c}_k z^{3k} = 3P(z)C(z^3),\end{aligned}$$

which yields the required result (4.2.16). ∎

Lemma 4.2.6 *Let $\boldsymbol{p} = \{p_j\} \in l_0$ denote a sequence satisfying the sum-rule condition of order at least $m \in \mathbb{N}$, and let P be the corresponding Laurent polynomial symbol as in (3.1.4), and therefore satisfying the formulation (3.1.31), (3.1.32) of Theorem 3.1.12. Also, for any integer $l \in \{1, \ldots, m\}$, let the sequence $\boldsymbol{a}_l = \{a_{l,j} : j \in \mathbb{Z}\} \in l_0$ be defined by*

$$\tfrac{1}{3}\sum_j a_{l,j} z^j := \frac{P(z)}{(1+z+z^2)^l}, \quad z \in \mathbb{C} \setminus \{0\}. \tag{4.2.17}$$

Then

$$\Delta^l\left(\mathcal{S}_{\boldsymbol{p}}^r \boldsymbol{c}\right) = \mathcal{S}_{\boldsymbol{a}_l}^r\left(\Delta^l \boldsymbol{c}\right), \quad r \in \mathbb{N}, \ \boldsymbol{c} \in l_0, \tag{4.2.18}$$

where, as in (2.2.14),

$$(\mathcal{S}_{\boldsymbol{a}_l}\boldsymbol{c})_j := \sum_k a_{l,j-3k}\boldsymbol{c}_k, \quad j \in \mathbb{Z}, \ \boldsymbol{c} \in l(\mathbb{Z}). \tag{4.2.19}$$

Proof.
Let $l \in \{1, \ldots, m\}$ be fixed, and define the Laurent polynomial

$$A_l(z) := \tfrac{1}{3}\sum_j a_{l,j} z^j, \quad z \in \mathbb{C} \setminus \{0\}. \tag{4.2.20}$$

It follows from (4.2.15) and (4.2.14) in Lemma 4.2.4, (4.2.16) in Lemma 4.2.5, the definition (4.2.17), (4.2.20), and finally (4.2.19), that, for any $\boldsymbol{c} \in l_0$,

$$\sum_j \left(\Delta^l\left(\mathcal{S}_{\boldsymbol{p}}\boldsymbol{c}\right)\right)_j z^j = (1-z)^l \left[\sum_j (\mathcal{S}_{\boldsymbol{p}}\boldsymbol{c})_j z^j\right]$$

$$
\begin{aligned}
&= 3\,(1-z)^l\,P(z)C(z^3) \\
&= 3\,(1-z)^l \left[\left(1+z+z^2\right)^l A_l(z)\right] C(z^3) \\
&= 3A_l(z)\left(1-z^3\right)^l C(z^3) \\
&= \sum_j a_{l,j} z^j \left[\sum_k \left(\Delta^l \boldsymbol{c}\right)_k (z^3)^k\right] \\
&= \sum_j a_{l,j-3k} z^{j-3k} \left[\sum_k \left(\Delta^l \boldsymbol{c}\right)_k (z^3)^k\right] \\
&= \sum_j \left[\sum_k a_{l,j-3k}\left(\Delta^l \boldsymbol{c}\right)_k\right] z^j = \sum_j \left(\mathcal{S}_{\boldsymbol{a}_l}\left(\Delta^l \boldsymbol{c}\right)\right)_j z^j,
\end{aligned}
$$

that is,

$$\Delta^l\left(\mathcal{S}_{\boldsymbol{p}}\boldsymbol{c}\right) = \mathcal{S}_{\boldsymbol{a}_l}\left(\Delta^l \boldsymbol{c}\right), \tag{4.2.21}$$

and thereby proving (4.2.18) for $r = 1$. Now let $r \geq 2$. It follows from (4.2.21) that

$$\Delta^l\left(\mathcal{S}_{\boldsymbol{p}}^r\boldsymbol{c}\right) = \Delta^l\left(\mathcal{S}_{\boldsymbol{p}}\left(\mathcal{S}_{\boldsymbol{p}}^{r-1}\boldsymbol{c}\right)\right) = \mathcal{S}_{\boldsymbol{a}_l}\left(\Delta^l\left(\mathcal{S}_{\boldsymbol{p}}^{r-1}\boldsymbol{c}\right)\right) = \cdots = \mathcal{S}_{\boldsymbol{a}_l}^r\left(\Delta^l \boldsymbol{c}\right),$$

and thereby completing our proof of (4.2.18). ■

We are now in a position to prove the following sufficient condition for the convergence of the cascade algorithm. The analogous result in binary subdivison can be found in [17] for a general integer $l \in \{1, \ldots, m\}$ (Theorem 3.7, p 48). The result here holds for $l = 2$. The proof below initially follows the same pattern as the proof in [17], with some non-trivial adaptations from (4.2.29) onwards.

Theorem 4.2.7 *Let $\boldsymbol{p} = \{p_j\} \in l_0$ be such that the sum-rule condition of order at least $m \geq 2$, as well as the interpolatory condition (3.1.2), are satisfied, with corresponding symbol P given by (3.1.4) and (3.1.31), (3.1.32). Also, let the sequence $\boldsymbol{a}_l = \{a_{l,j} : j \in \mathbb{Z}\} \in l_0$ be defined as in (4.2.17), (4.2.20). Suppose, moreover, that the subdivision operator $\mathcal{S}_{\boldsymbol{a}_2}$, as defined in (4.2.19), is contractive, in the sense that*

$$||\mathcal{S}_{\boldsymbol{a}_2}||_\infty < 1, \tag{4.2.22}$$

where, as in (4.2.3) of Theorem 4.2.2,

$$||\mathcal{S}_{\boldsymbol{a}_2}||_\infty = \max\left\{\sum_j |a_{2,3j}|, \sum_j |a_{2,3j+1}|, \sum_j |a_{2,3j+2}|\right\}. \tag{4.2.23}$$

Then the cascade algorithm based on $\boldsymbol{p}$, as given in (4.1.7), is convergent.

Proof.
Fix $x \in \mathbb{R}$ and $r \in \{0, 1, \ldots\}$, and let $\boldsymbol{\delta} = \{\delta_j\}$ denote the Kronecker delta sequence, as in (2.2.13). By applying (4.1.11) in Theorem 4.1.4, (2.2.18) and (2.2.14), we obtain

$$\begin{aligned} h_{r+1}(x) - h_r(x) &= \sum_j p_j^{[r+1]} h(3^{r+1}x - j) - \sum_j p_j^{[r]} h(3^r x - j) \\ &= \sum_j \left[\sum_k p_{j-3k} p_k^{[r]} \right] h(3^{r+1}x - j) - \sum_j p_j^{[r]} h(3^r x - j), \end{aligned} \tag{4.2.24}$$

with h denoting the hat function as in (2.1.6) in Example 2.1.4, where it was also shown that h is a refinable function with refinement sequence $\{\tilde{p}_j\}$ given by

$$\{\tilde{p}_{-2}, \tilde{p}_{-1}, \tilde{p}_0, \tilde{p}_1, \tilde{p}_2\} = \left\{\tfrac{1}{3}, \tfrac{2}{3}, 1, \tfrac{2}{3}, \tfrac{1}{3}\right\}; \quad \tilde{p}_j = 0, \ j \notin \{-2, -1, 0, 1, 2\}. \tag{4.2.25}$$

It follows from the refinability of h with respect to the refinement sequence $\{\tilde{p}_j\}$ in (4.2.25) that

$$\begin{aligned} \sum_j p_j^{[r]} h(3^r x - j) &= \sum_j p_j^{[r]} \left[\sum_k \tilde{p}_k h(3^{r+1}x - 3j - k) \right] \\ &= \sum_j p_j^{[r]} \left[\sum_k \tilde{p}_{k-3j} h(3^{r+1}x - k) \right] \\ &= \sum_k \left[\sum_j \tilde{p}_{k-3j} p_j^{[r]} \right] h(3^{r+1}x - k) \\ &= \sum_j \left[\sum_k \tilde{p}_{j-3k} p_k^{[r]} \right] h(3^{r+1}x - j), \end{aligned}$$

so that (4.2.24) becomes

$$\begin{aligned} h_{r+1}(x) - h_r(x) &= \sum_j \left[\sum_k (p_{j-3k} - \tilde{p}_{j-3k})\, p_k^{[r]} \right] h(3^{r+1}x - j) \\ &= \sum_j \left[\sum_k (p_{3j-3k} - \tilde{p}_{3j-3k})\, p_k^{[r]} \right] h(3^{r+1}x - 3j) \\ &\quad + \sum_j \left[\sum_k (p_{3j+1-3k} - \tilde{p}_{3j+1-3k})\, p_k^{[r]} \right] h(3^{r+1}x - 3j - 1) \\ &\quad + \sum_j \left[\sum_k (p_{3j+2-3k} - \tilde{p}_{3j+2-3k})\, p_k^{[r]} \right] h(3^{r+1}x - 3j - 2). \end{aligned} \tag{4.2.26}$$

After observing from (4.2.25) that

$$\tilde{p}_{3j} = \delta_j, \quad j \in \mathbb{Z},$$

it follows from (4.2.26) and (3.1.2) that

$$\begin{aligned}
&h_{r+1}(x) - h_r(x) \\
&= \sum_j \left[\sum_k \left(p_{3j+1-3k} - \tilde{p}_{3j+1-3k}\right) p_k^{[r]} \right] h(3^{r+1}x - 3j - 1) \\
&\quad + \sum_j \left[\sum_k \left(p_{3j+2-3k} - \tilde{p}_{3j+2-3k}\right) p_k^{[r]} \right] h(3^{r+1}x - 3j - 2) \\
&= \sum_j \left[\sum_k p_{3j+1-3k} p_k^{[r]} - \sum_k \tilde{p}_{3j+1-3k} p_k^{[r]} \right] h(3^{r+1}x - 3j - 1) \\
&\quad + \sum_j \left[\sum_k p_{3j+2-3k} p_k^{[r]} - \sum_k \tilde{p}_{3j+2-3k} p_k^{[r]} \right] h(3^{r+1}x - 3j - 2). \qquad (4.2.27)
\end{aligned}$$

Moreover, by virtue of Theorem 3.1.2 and the fact that the interpolatory condition (3.1.2) holds, the subdivision operator $\mathcal{S}_{\boldsymbol{p}}$ is interpolatory, and thus, from (2.2.18) and (2.2.16),

$$p_{3j}^{[r+1]} = p_j^{[r]}, \quad j \in \mathbb{Z}. \qquad (4.2.28)$$

By applying (2.2.14), (2.2.18) and (4.2.28) in (4.2.27), we obtain

$$\begin{aligned}
h_{r+1}(x) - h_r(x) &= \sum_j \left[p_{3j+1}^{[r+1]} - \sum_k \tilde{p}_{3j+1-3k} p_{3k}^{[r+1]} \right] h(3^{r+1}x - 3j - 1) \\
&\quad + \sum_j \left[p_{3j+2}^{[r+1]} - \sum_k \tilde{p}_{3j+2-3k} p_{3k}^{[r+1]} \right] h(3^{r+1}x - 3j - 2).
\end{aligned} \qquad (4.2.29)$$

Next, we use (4.2.25) and (2.4.3) to deduce, with the notation $\boldsymbol{p}^{[r]} = \left\{p_j^{[r]} : j \in \mathbb{Z}\right\}$, that, for any $j \in \mathbb{Z}$,

$$\begin{aligned}
&p_{3j+1}^{[r+1]} - \sum_k \tilde{p}_{3j+1-3k} p_{3k}^{[r+1]} \\
&= p_{3j+1}^{[r+1]} - \sum_k \tilde{p}_{3k+1} p_{3j-3k}^{[r+1]} \\
&= p_{3j+1}^{[r+1]} - \tfrac{1}{3} p_{3j+3}^{[r+1]} - \tfrac{2}{3} p_{3j}^{[r+1]} \\
&= \left(-\tfrac{1}{3}\right) \left[p_{3j+3}^{[r+1]} - 3p_{3j+1}^{[r+1]} + 2p_{3j}^{[r+1]} \right]
\end{aligned}$$

$$= \left(-\tfrac{1}{3}\right)\left[\left(p^{[r+1]}_{3j+3} - 2p^{[r+1]}_{3j+2} + p^{[r+1]}_{3j+1}\right) + 2\left(p^{[r+1]}_{3j+2} - 2p^{[r+1]}_{3j+1} + p^{[r+1]}_{3j}\right)\right]$$
$$= \left(-\tfrac{1}{3}\right)\left[\left(\Delta^2\boldsymbol{p}^{[r+1]}\right)_{3j+3} + 2\left(\Delta^2\boldsymbol{p}^{[r+1]}\right)_{3j+2}\right], \qquad (4.2.30)$$

and, similarly,

$$p^{[r+1]}_{3j+2} - \sum_k \tilde{p}_{3j+2-3k} p^{[r+1]}_{3k}$$
$$= p^{[r+1]}_{3j+2} - \sum_k \tilde{p}_{3k+2} p^{[r+1]}_{3j-3k}$$
$$= p^{[r+1]}_{3j+2} - \tfrac{2}{3}p^{[r+1]}_{3j+3} - \tfrac{1}{3}p^{[r+1]}_{3j}$$
$$= \left(-\tfrac{1}{3}\right)\left[2p^{[r+1]}_{3j+3} - 3p^{[r+1]}_{3j+2} + p^{[r+1]}_{3j}\right]$$
$$= \left(-\tfrac{1}{3}\right)\left[2\left(p^{[r+1]}_{3j+3} - 2p^{[r+1]}_{3j+2} + p^{[r+1]}_{3j+1}\right) + \left(p^{[r+1]}_{3j+2} - 2p^{[r+1]}_{3j+1} + p^{[r+1]}_{3j}\right)\right]$$
$$= \left(-\tfrac{1}{3}\right)\left[2\left(\Delta^2\boldsymbol{p}^{[r+1]}\right)_{3j+3} + \left(\Delta^2\boldsymbol{p}^{[r+1]}\right)_{3j+2}\right]. \qquad (4.2.31)$$

By applying (4.2.30) and (4.2.31) in (4.2.29), we obtain

$$h_{r+1}(x) - h_r(x)$$
$$= \left(-\tfrac{1}{3}\right)\sum_j \left[\left(\Delta^2\boldsymbol{p}^{[r+1]}\right)_{3j+3} + 2\left(\Delta^2\boldsymbol{p}^{[r+1]}\right)_{3j+2}\right] h(3^{r+1}x - 3j - 1)$$
$$+ \left(-\tfrac{1}{3}\right)\sum_j \left[2\left(\Delta^2\boldsymbol{p}^{[r+1]}\right)_{3j+3} + \left(\Delta^2\boldsymbol{p}^{[r+1]}\right)_{3j+2}\right] h(3^{r+1}x - 3j - 2). \qquad (4.2.32)$$

Now denote by k the (unique) integer for which

$$\tfrac{k}{3^r} \leq x < \tfrac{k+1}{3^r},$$

from which it follows, together with the fact that (2.1.6) yields

$$h(x) = 0, \quad x \notin (-1, 1),$$

as well as (4.2.32), that

$$h_{r+1}(x) - h_r(x)$$
$$= \left(-\tfrac{1}{3}\right)\left[\left(\Delta^2\boldsymbol{p}^{[r+1]}\right)_{3k+3} + 2\left(\Delta^2\boldsymbol{p}^{[r+1]}\right)_{3k+2}\right] h(3^{r+1}x - 3k - 1)$$
$$+ \left(-\tfrac{1}{3}\right)\left[2\left(\Delta^2\boldsymbol{p}^{[r+1]}\right)_{3k+3} + \left(\Delta^2\boldsymbol{p}^{[r+1]}\right)_{3k+2}\right] h(3^{r+1}x - 3k - 2),$$

and thus, since also (2.1.6) yields $||h||_\infty = 1$, we have

$$|h_{r+1}(x) - h_r(x)| \leq \tfrac{1}{3}\left(6||\Delta^2\boldsymbol{p}^{[r+1]}||_\infty\right)||h||_\infty = 2||\Delta^2\boldsymbol{p}^{[r+1]}||_\infty. \qquad (4.2.33)$$

Next, we observe, from (2.2.18), together with (4.2.18) in Lemma 4.2.6, that

$$\Delta^2 \boldsymbol{p}^{[r+1]} = \Delta^2 \left(\mathcal{S}_{\boldsymbol{p}}^{r+1} \boldsymbol{\delta} \right) = \mathcal{S}_{\boldsymbol{a}_2}^{r+1} \left(\Delta^2 \boldsymbol{\delta} \right). \tag{4.2.34}$$

With the definition

$$\rho_2 := \rho_{\boldsymbol{a}_2} = ||\mathcal{S}_{\boldsymbol{a}_2}||_\infty, \tag{4.2.35}$$

from (4.2.3) in Theorem 4.2.2, we now repeatedly apply the inequality (4.2.4) in Theorem 4.2.2 to obtain

$$\begin{aligned} ||\mathcal{S}_{\boldsymbol{a}_2}^{r+1} \left(\Delta^2 \boldsymbol{\delta} \right) ||_\infty = ||\mathcal{S}_{\boldsymbol{a}_2} \left(\mathcal{S}_{\boldsymbol{a}_2}^{r} \left(\Delta^2 \boldsymbol{\delta} \right) \right) ||_\infty &\leq \rho_2 ||\mathcal{S}_{\boldsymbol{a}_2}^{r} \left(\Delta^2 \boldsymbol{\delta} \right) ||_\infty \\ &\leq \cdots \leq \left(\rho_2\right)^{r+1} ||\Delta^2 \boldsymbol{\delta}||_\infty. \end{aligned} \tag{4.2.36}$$

Moreover, for any $j \in \mathbb{Z}$, it follows from (4.2.13) in Lemma 4.2.3, together with (2.2.13), that

$$\left(\Delta^2 \boldsymbol{\delta}\right)_j = \sum_{k=0}^{2} (-1)^k \tbinom{2}{k} \delta_{j-k} = (-1)^j \tbinom{2}{j},$$

so that

$$||\Delta^2 \boldsymbol{\delta}||_\infty = \max \left\{ \tbinom{2}{j} : j = 0, 1, 2 \right\} = 2,$$

which, together with (4.2.34) and (4.2.36), yields the inequality

$$||\Delta^2 \boldsymbol{p}^{r+1}||_\infty \leq 2 \left(\rho_2\right)^{r+1}. \tag{4.2.37}$$

It follows from (4.2.33) and (4.2.37) that

$$|h_{r+1}(x) - h_r(x)| \leq 4 \left(\rho_2\right)^{r+1},$$

and thus

$$||h_{r+1} - h_r||_\infty \leq 4 \left(\rho_2\right)^{r+1}. \tag{4.2.38}$$

Now let $j, k \in \mathbb{N}$, with $j < k$. It follows, from (4.2.38), that

$$\begin{aligned} ||h_k - h_j||_\infty = \left|\left| \sum_{r=j}^{k-1} \left(h_{r+1} - h_r\right) \right|\right|_\infty &\leq \sum_{r=j}^{k-1} ||h_{r+1} - h_r||_\infty \\ &\leq \sum_{r=j}^{k-1} 4 \left(\rho_2\right)^{r+1} \\ &= 4 \left(\rho_2\right)^{j+1} \sum_{r=j}^{k-1} \left(\rho_2\right)^{r-j} \\ &= 4 \left(\rho_2\right)^{j+1} \sum_{r=0}^{k-1-j} \left(\rho_2\right)^{r} \end{aligned}$$

$$= 4\,(\rho_2)^{j+1}\,\frac{1-(\rho_2)^{k-j}}{1-\rho_2}$$
$$< \frac{4}{1-\rho_2}\,(\rho_2)^{j+1}, \qquad (4.2.39)$$

after having noted also from (4.2.35) and (4.2.22) that

$$\rho_2 \in (0,1). \qquad (4.2.40)$$

Let $\varepsilon > 0$. It then follows from (4.2.39), together with (4.2.40), that there exists an integer $R(\varepsilon) \in \mathbb{N}$ such that, for $k > j > R(\varepsilon)$, we have

$$||h_k - h_j||_\infty < \varepsilon. \qquad (4.2.41)$$

It follows that

$$\max_{\mu/2 \le x \le \nu/2} |h_k(x) - h_j(x)| < \varepsilon$$

for $k > j > R(\varepsilon)$, according to which $\{h_j : j = 0, 1, \ldots\}$ is a Cauchy sequence with respect to the maximum norm in $C\left[\frac{\mu}{2}, \frac{\nu}{2}\right]$. According to a standard result in analysis (see e.g. [16], Example 2.2-5, p 61), $C\left[\frac{\mu}{2}, \frac{\nu}{2}\right]$ is a complete normed linear space (or Banach space) with respect to the maximum norm on $C\left[\frac{\mu}{2}, \frac{\nu}{2}\right]$. We therefore deduce that there exists a function $h_{\boldsymbol{p}} \in C\left[\frac{\mu}{2}, \frac{\nu}{2}\right]$ such that

$$\max_{\mu/2 \le x \le \nu/2} |h_{\boldsymbol{p}}(x) - h_r(x)| \to 0, \quad r \to \infty. \qquad (4.2.42)$$

Now observe from (4.2.42) and (4.1.8) in Theorem 4.1.4 that

$$h_{\boldsymbol{p}}(\tfrac{\mu}{2}) = h_{\boldsymbol{p}}(\tfrac{\nu}{2}) = 0.$$

Hence, with the definition

$$h_{\boldsymbol{p}}(x) := 0, \quad x \notin \left[\tfrac{\mu}{2}, \tfrac{\nu}{2}\right],$$

we deduce from (4.2.42) and (4.1.8) in Theorem 4.1.4 that

$$||h_{\boldsymbol{p}} - h_r||_\infty := \sup_x |h_{\boldsymbol{p}}(x) - h_r(x)| \to 0, \quad r \to \infty,$$

which shows that the cascade algorithm (4.1.7) is convergent, with limit function $h_{\boldsymbol{p}}$, and thereby completing our proof. ■

We end this section by combining Theorem 3.1.1, Corollary 3.2.3 and Theorems 4.1.7 and 4.2.7 to obtain the following conclusion, completing the argument of Chapters 3 and 4.

Corollary 4.2.8 *For any integer $m \geq 2$, let the sequence $\boldsymbol{p}_m = \{p_{m,j} : j \in \mathbb{Z}\} \in l_0$ be defined by*

$$\tfrac{1}{3}\sum_j p_{m,j} z^j = P_m(z) := \left(\frac{1+z+z^2}{3}\right)^m z^{-d_m} U_m(z), \tag{4.2.43}$$

with U_m defined by (3.2.29), (3.2.30), (3.2.26), and d_m given by (3.2.3), and suppose that the sequence $\{a_{m,j} : j \in \mathbb{Z}\}$ defined by

$$\tfrac{1}{3}\sum_j a_{m,j} z^j = A_m(z) := \left(\tfrac{1}{3}\right)^m (1+z+z^2)^{m-2} z^{-d_m} U_m(z), \tag{4.2.44}$$

satisfies the condition

$$\max\left\{\sum_j |a_{m,3j}|, \sum_j |a_{m,3j+1}|, \sum_j |a_{m,3j+2}|\right\} < 1. \tag{4.2.45}$$

Then the subdivision operator $\mathcal{S}_m := \mathcal{S}_{\boldsymbol{p}_m}$ provides a convergent interpolatory subdivision scheme, with limit function $\phi_m := \phi_{\boldsymbol{p}_m}$ such that

$$\phi_m(j) = \delta_j, \quad j \in \mathbb{Z}. \tag{4.2.46}$$

Chapter 5

Ternary interpolatory subdivision schemes

We proceed to apply the results obtained in Chapters 2, 3 and 4 in specific examples.

5.1 Rendering closed curves

We start by constructing an algorithm which will enable us to render closed curves, illustrating the interpolatory subdivision scheme based on a refinement sequence as in Corollary 4.2.8.

We will rely on the following lemma, the proof of which is a straightforward generalisation of the proof for the 2-refinability case (see [1], Lemma 3.3.1, p 95).

Lemma 5.1.1 *Let $\boldsymbol{c} = \{\boldsymbol{c}_j\} \in l(\mathbb{Z})$ denote a sequence of control points in $\mathbb{R}^s$ for $s \geq 1$, such that $\Delta^k \boldsymbol{c} \in l^\infty$ for $k = 1$ or $k = 2$. Suppose, for $M \in \mathbb{N}$, that $\boldsymbol{c}$ satisfies the periodicity condition*

$$\boldsymbol{c}_{j+M+1} = \boldsymbol{c}_j, \quad j \in \mathbb{Z}. \tag{5.1.1}$$

Then the sequences $\boldsymbol{c}^r := \left\{\boldsymbol{c}^r_j\right\}$, generated recursively by the subdivision scheme (2.2.4), are also periodic, that is,

$$\boldsymbol{c}^r_{j+3^r(M+1)} = \boldsymbol{c}^r_j, \quad j \in \mathbb{Z}. \tag{5.1.2}$$

Moreover, the limit curve $\boldsymbol{F}_{\boldsymbol{c}}$ in (2.2.1) is also periodic, with

$$\boldsymbol{F}_{\boldsymbol{c}}(t + M + 1) = \boldsymbol{F}_{\boldsymbol{c}}(t), \tag{5.1.3}$$

and hence it is closed.

We consider an arbitrarily ordered set $\{\boldsymbol{c}_0, \ldots, \boldsymbol{c}_M\}$ with $\boldsymbol{c}_0 \neq \boldsymbol{c}_M$, where $M \in \mathbb{N}$. We extend this set periodically according to (5.1.1). Then, if ϕ is an interpolatory scaling function with refinement sequence $\{p_j\}$, it follows from Lemma 5.1.1 that we can apply the interpolatory subdivision scheme described in (2.2.10) to render closed curves $\boldsymbol{F_c}$ according to (2.2.1), provided that the subdivision scheme converges.

The algorithm below is an adaptation of an algorithm based on 2-refinable functions (see [1], Algorithm 3.3.2, p 104).

Algorithm 5.1.2 For rendering closed curves.

Let $\{p_j\}$ denote a refinement sequence with $\operatorname{supp}\{p_j\} = [-d, d]\big|_{\mathbb{Z}}$ for an integer $d \in \mathbb{N}$. Let the weight sequences $\{w_j^2\}$ and $\{w_j^3\}$ be given by (2.2.5).

(1) User to arbitrarily input an ordered set of control points $\boldsymbol{c}_0, \ldots, \boldsymbol{c}_M$, with $\boldsymbol{c}_0 \neq \boldsymbol{c}_M$.

(2) Initialization: Relabel $\boldsymbol{c}_j^0 := \boldsymbol{c}_j$, $j = 0, \ldots, M$. Set

$$\begin{cases} \boldsymbol{c}_j^0 := \boldsymbol{c}_{M+j+1}^0, & j = -\lfloor \frac{d}{3} \rfloor, \ldots, -1; \\ \boldsymbol{c}_{M+j}^0 := \boldsymbol{c}_{j-1}^0, & j = 1, \ldots, 1 + \lfloor \frac{d-1}{3} \rfloor. \end{cases}$$

(3) For $r = 1, 2, \ldots,$ compute

$$\begin{cases} \boldsymbol{c}_{3n}^r := \boldsymbol{c}_n^{r-1}, & n = 0, \ldots, 3^{r-1}(M+1) - 1; \\ \boldsymbol{c}_{3n-1}^r := \displaystyle\sum_{k=-\lfloor (d-1)/3 \rfloor}^{\lfloor (d+1)/3 \rfloor} w_k^2 \boldsymbol{c}_{n-k}^{r-1}, & n = 1, \ldots, 3^{r-1}(M+1); \\ \boldsymbol{c}_{3n-2}^r := \displaystyle\sum_{k=-\lfloor (d-2)/3 \rfloor}^{\lfloor (d+2)/3 \rfloor} w_k^3 \boldsymbol{c}_{n-k}^{r-1}, & n = 1, \ldots, 3^{r-1}(M+1), \end{cases}$$

where, for $r \geq 2$,

$$\begin{cases} \boldsymbol{c}_j^{r-1} := \boldsymbol{c}_{3^{r-1}(M+1)+j}^{r-1}, & j = -\lfloor \frac{d}{3} \rfloor, \ldots, -1; \\ \boldsymbol{c}_{3^{r-1}(M+1)-1+j}^{r-1} := \boldsymbol{c}_{j-1}^{r-1}, & j = 1, \ldots, 1 + \lfloor \frac{d-1}{3} \rfloor. \end{cases}$$

(4) Stop when $r = r_0$, where $2^{r_0}/\sqrt{2}$ does not exceed the maximum of the number of horizontal pixels and the number of vertical pixels of the display monitor.

(5) User to manipulate the control points by moving one or more of them, inserting additional ones (while keeping track of the ordering), or removing a desirable number of them. Repeat Steps 1 through 4.

5.2 Examples

Example 5.2.1 Let $m = 2$, so that, as in Table 3.1, we have

$$U_2(z) = 1. \tag{5.2.1}$$

Observe that $U_2 \in \pi_0$ and that U_2 is a symmetric polynomial, as expected from Theorem 3.3.2. By applying (5.2.1) in (4.2.43) in Corollary 4.2.8, with, from (3.2.3), $d_2 = 2$, we obtain

$$\sum_j p_{2,j} z^j = \tfrac{1}{3}\left(1 + z + z^2\right)^2 \left(z^{-2}\right)(1) \tag{5.2.2}$$

$$= \tfrac{1}{3}\left(z^{-2}\right)\left(1 + 2z + 3z^2 + 2z^3 + z^4\right), \tag{5.2.3}$$

and thus

$$\{p_{2,-2}, \ldots, p_{2,2}\} = \left\{\tfrac{1}{3}, \tfrac{2}{3}, 1, \tfrac{2}{3}, \tfrac{1}{3}\right\}, \quad p_{2,j} = 0, \; j \notin \{-2, \ldots, 2\}. \tag{5.2.4}$$

We note that the refinement sequence $\{p_{2,j}\}$ in (5.2.4) is a symmetric sequence, and satisfies the interpolatory condition

$$p_{2,3j} = \delta_j, \quad j \in \mathbb{Z}, \tag{5.2.5}$$

as expected from Theorem 3.3.3. It follows from (5.2.2) and (3.1.4) that

$$P_2(z) = \tfrac{1}{9}\left(1 + z + z^2\right)^2 \left(z^{-2}\right)(1), \tag{5.2.6}$$

so that, from (4.2.44),

$$A_2(z) = \tfrac{1}{3}\sum_j a_{2,j} z^j = \tfrac{1}{9}\left(z^{-2}\right), \tag{5.2.7}$$

and thus

$$\{a_{2,-2}\} = \left\{\tfrac{1}{3}\right\}, \quad a_{2,j} = 0, \; j \neq -2. \tag{5.2.8}$$

By using (5.2.8) in (4.2.45), we see that

$$\max\left\{\tfrac{1}{3}\right\} = \tfrac{1}{3} < 1, \tag{5.2.9}$$

so that it follows from Corollary 4.2.8 that the subdivision operator $\mathcal{S}_2$ provides a convergent subdivision scheme, with interpolatory refinable limit function ϕ_2. Moreover, we may apply Corollary 2.4.15 to deduce that ϕ_2 provides a partition of unity, ϕ_2 is the only function in C_0 that is refinable and provides a partition of unity, and ϕ_2 is a symmetric function, since $\{p_{2,j}\}$ is a symmetric sequence.

In Figure 5.1, we display the generated points for a given set of control points, with $M = 12$ in Lemma 5.1.1, for seven iterations, for the ternary interpolatory scheme obtained here, as well as for the analogous Dubuc-Deslauriers binary interpolatory scheme, as introduced in [11] and [10] (see also [1], Section 8.2, pp 302-312). The figures in (a) are obtained by using Algorithm 5.1.2, while the figures in (b) are obtained by using an adaptation of Algorithm 3.3.1, p 97, in [1], as discussed on pp 312-313 in [1]. Observe that convergence is faster for the ternary scheme than for the binary scheme; this is indeed the main advantage of using ternary schemes rather than binary schemes. However, the trade-off is that the ternary refinement sequence is "longer" than the analogous binary refinement sequence (see Table 8.2.1, p 305 in [1]); as a result, the algorithm for rendering a closed curve takes a longer time to execute. Observe that the limit curve passes through the initial control points in both cases, as required of interpolatory subdivision schemes. ■

Example 5.2.2 Next, let $m = 4$, so that, as in Table 3.1, we have

$$U_4(z) = \tfrac{1}{3}\left(-4 + 11z - 4z^2\right). \tag{5.2.10}$$

Observe that $U_4 \in \pi_2$ and that U_4 is a symmetric polynomial, as expected from Theorem 3.3.2. By applying (5.2.10) in (4.2.43) in Corollary 4.2.8, with, from (3.2.3), $d_4 = 5$, we obtain

$$\sum_j p_{4,j} z^j = \tfrac{1}{27}\left(1 + z + z^2\right)^4 \left(z^{-5}\right)\left(\tfrac{1}{3}\right)\left(-4 + 11z - 4z^2\right) \tag{5.2.11}$$

$$= \tfrac{1}{81}\left(z^{-5}\right)\left(-4 - 5z + 30z^3 + 60z^4 + 81z^5 + 60z^6 + 30z^7 - 5z^9 - 4z^{10}\right), \tag{5.2.12}$$

and thus

$$\{p_{4,-5}, \ldots, p_{4,5}\} = \left\{-\tfrac{4}{81}, -\tfrac{5}{81}, 0, \tfrac{10}{27}, \tfrac{20}{27}, 1, \tfrac{20}{27}, \tfrac{10}{27}, 0, -\tfrac{5}{81}, -\tfrac{4}{81}\right\},$$
$$p_{4,j} = 0, \; j \notin \{-5, \ldots, 5\}. \tag{5.2.13}$$

We note that the refinement sequence $\{p_{4,j}\}$ in (5.2.13) is a symmetric sequence, and satisfies the interpolatory condition

$$p_{4,3j} = \delta_j, \quad j \in \mathbb{Z}, \tag{5.2.14}$$

as expected from Theorem 3.3.3. It follows from (5.2.11) and (3.1.4) that

$$P_4(z) = \tfrac{1}{81}\left(1 + z + z^2\right)^4 \left(z^{-5}\right)\left(\tfrac{1}{3}\right)\left(-4 + 11z - 4z^2\right), \tag{5.2.15}$$

so that, from (4.2.44),

$$\begin{aligned} A_4(z) = \tfrac{1}{3}\sum_j a_{4,j}z^j &= \tfrac{1}{81}\left(1+z+z^2\right)^2\left(z^{-5}\right)\left(\tfrac{1}{3}\right)\left(-4+11z-4z^2\right) \\ &= \tfrac{1}{243}\left(z^{-5}\right)\left(-4+3z+6z^2+17z^3+6z^4+3z^5-4z^6\right), \end{aligned} \tag{5.2.16}$$

and thus

$$\{a_{4,-5},\ldots,a_{4,1}\} = \left\{-\tfrac{4}{81},\tfrac{1}{27},\tfrac{2}{27},\tfrac{17}{81},\tfrac{2}{27},\tfrac{1}{27},-\tfrac{4}{81}\right\}, \quad a_{4,j}=0,\ j\notin\{-5,\ldots,1\}. \tag{5.2.17}$$

By using (5.2.17) in (4.2.45), we see that

$$\max\left\{\tfrac{1}{9},\tfrac{25}{81},\tfrac{1}{9}\right\} = \tfrac{25}{81} < 1, \tag{5.2.18}$$

so that it follows from Corollary 4.2.8 that the subdivision operator $\mathcal{S}_4$ provides a convergent subdivision scheme, with interpolatory refinable limit function ϕ_4. Moreover, we may apply Corollary 2.4.15 to deduce that ϕ_4 provides a partition of unity, ϕ_4 is the only function in C_0 that is refinable and provides a partition of unity, and ϕ_4 is a symmetric function, since $\{p_{4,j}\}$ is a symmetric sequence.

In Figure 5.2, we display the generated points for a given set of control points, with $M = 12$ in Lemma 5.1.1, for seven iterations, for the ternary interpolatory scheme obtained here, as well as for the analogous Dubuc-Deslauriers binary interpolatory scheme, as introduced in [11] and [10] (see also [1], Section 8.2, pp 302-312). Again, observe that convergence is faster for the ternary scheme than for the binary scheme, at the expense of a "longer" refinement sequence, and that the limit curve passes through the initial control points in both cases, as required of interpolatory subdivision schemes. ■

Example 5.2.3 Lastly, let $m = 10$, so that, as in Table 3.1, we have

$$\begin{aligned} U_{10}(z) = \tfrac{1}{81}\big(&715-6380z+24475z^2-52190z^3+66841z^4 \\ &-52190z^5+24475z^6-6380z^7+715z^8\big). \end{aligned} \tag{5.2.19}$$

Observe that $U_{10} \in \pi_8$ and that U_{10} is a symmetric polynomial, as expected from Theorem 3.3.2. By applying (5.2.19) in (4.2.43) in Corollary 4.2.8, with, from (3.2.3), $d_{10} = 14$, we obtain

$$\begin{aligned} \sum_j p_{10,j}z^j &= \tfrac{1}{19683}\left(1+z+z^2\right)^{10}\left(z^{-14}\right)\left(\tfrac{1}{81}\right)\big(715-6380z+24475z^2-52190z^3 \\ &\quad +66841z^4-52190z^5+24475z^6-6380z^7+715z^8\big) \qquad (5.2.20) \\ &= \tfrac{897}{2000000}z^{-14}+\tfrac{483}{1000000}z^{-13}-\tfrac{5137}{1000000}z^{-11}-\tfrac{5651}{1000000}z^{-10}+\tfrac{113}{4000}z^{-8} \end{aligned}$$

$$\begin{aligned}&+\tfrac{3229}{100000}z^{-7}-\tfrac{211}{2000}z^{-5}-\tfrac{659}{5000}z^{-4}+\tfrac{791}{2000}z^{-2}+\tfrac{7911}{10000}z^{-1}+1\\&+\tfrac{7911}{10000}z^{1}+\tfrac{791}{2000}z^{2}-\tfrac{659}{5000}z^{4}-\tfrac{211}{2000}z^{5}+\tfrac{3229}{100000}z^{7}+\tfrac{113}{4000}z^{8}\\&-\tfrac{5651}{1000000}z^{10}-\tfrac{5137}{1000000}z^{11}+\tfrac{483}{1000000}z^{13}+\tfrac{897}{2000000}z^{14},\end{aligned}\tag{5.2.21}$$

and thus

$$\begin{aligned}\{p_{10,-14},\ldots,p_{10,14}\}=\Big\{&\tfrac{897}{2000000},\tfrac{483}{1000000},0,-\tfrac{5137}{1000000},-\tfrac{5651}{1000000},0,\tfrac{113}{4000},\tfrac{3229}{100000},\\&0,-\tfrac{211}{2000},-\tfrac{659}{5000},0,\tfrac{791}{2000},\tfrac{7911}{10000},1,\tfrac{7911}{10000},\tfrac{791}{2000},0,-\tfrac{659}{5000},-\tfrac{211}{2000},0,\tfrac{3229}{100000},\tfrac{113}{4000},\\&0,-\tfrac{5651}{1000000},-\tfrac{5137}{1000000},0,\tfrac{483}{1000000},\tfrac{897}{2000000}\Big\},p_{10,j}=0,\ j\notin\{-14,\ldots,14\}.\end{aligned}\tag{5.2.22}$$

We note that the refinement sequence $\{p_{10,j}\}$ in (5.2.22) is a symmetric sequence, and satisfies the interpolatory condition

$$p_{10,3j}=\delta_j,\quad j\in\mathbb{Z},\tag{5.2.23}$$

as expected from Theorem 3.3.3. It follows from (5.2.20) and (3.1.4) that

$$P_{10}(z)=\tfrac{1}{59049}\left(1+z+z^2\right)^{10}\left(z^{-14}\right)U_{10}(z),\tag{5.2.24}$$

with U_{10} given by (5.2.19), so that, from (4.2.44),

$$\begin{aligned}A_{10}(z)&=\tfrac{1}{3}\sum_j a_{10,j}z^j\\&=\tfrac{1}{59049}\left(1+z+z^2\right)^{8}\left(z^{-14}\right)U_{10}(z)\\&=\tfrac{1}{4782969}\left(z^{-14}\right)\big(715-660z-825z^2-5990z^3+6051z^4+8178z^5\\&\quad-23340z^6-25850z^7-40730z^8-64020z^9+68380z^{10}+162600z^{11}\\&\quad+269000z^{12}+162600z^{13}+68380z^{14}-64020z^{15}-40730z^{16}\\&\quad-25850z^{17}-23340z^{18}+8178z^{19}+6051z^{20}-5990z^{21}-825z^{22}\\&\quad-660z^{23}+715z^{24}\big),\end{aligned}\tag{5.2.25}$$

and thus

$$\begin{aligned}\{a_{10,-14},\ldots,a_{10,10}\}=\Big\{&\tfrac{715}{1594323},-\tfrac{220}{531441},-\tfrac{275}{531441},-\tfrac{5990}{1594323},\tfrac{2017}{531441},\tfrac{2726}{531441},-\tfrac{7780}{531441},\\&-\tfrac{25850}{1594323},-\tfrac{40730}{1594323},-\tfrac{21340}{531441},\tfrac{68380}{1594323},\tfrac{54200}{531441},\tfrac{269000}{1594323},\tfrac{54200}{531441},\tfrac{68380}{1594323},-\tfrac{21340}{531441},\\&-\tfrac{40730}{1594323},-\tfrac{25850}{1594323},-\tfrac{7780}{531441},\tfrac{2726}{531441},\tfrac{2017}{531441},-\tfrac{5990}{1594323},-\tfrac{275}{531441},-\tfrac{220}{531441},\tfrac{715}{1594323}\Big\},\\&a_{10,j}=0,\ j\notin\{-14,\ldots,10\}.\end{aligned}\tag{5.2.26}$$

By using (5.2.26) in (4.2.45), we see that

$$\max\left\{\tfrac{313274}{1594323},\tfrac{457130}{1594323},\tfrac{313274}{1594323}\right\}=\tfrac{457130}{1594323}<1,\tag{5.2.27}$$

so that it follows from Corollary 4.2.8 that the subdivision operator $\mathcal{S}_{10}$ provides a convergent subdivision scheme, with interpolatory refinable limit function ϕ_{10}. Moreover, we may apply Corollary 2.4.15 to deduce that ϕ_{10} provides a partition of unity, ϕ_{10} is the only function in C_0 that is refinable and provides a partition of unity, and ϕ_{10} is a symmetric function, since $\{p_{10,j}\}$ is a symmetric sequence.

In Figure 5.3, we display the generated points for a given set of control points, with $M = 12$ in Lemma 5.1.1, for seven iterations, for the ternary interpolatory scheme obtained here, as well as for the analogous Dubuc-Deslauriers binary interpolatory scheme, as introduced in [11] and [10] (see also [1], Section 8.2, pp 302-312). Again, observe that convergence is faster for the ternary scheme than for the binary scheme, at the expense of a "longer" refinement sequence, and that the limit curve passes through the initial control points in both cases, as required of interpolatory subdivision schemes. ■

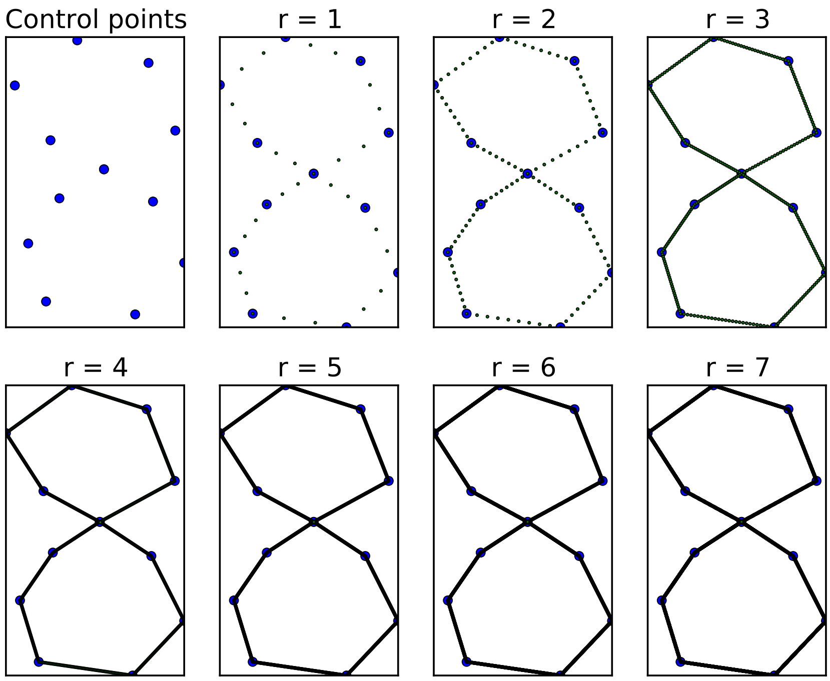
We end this section with convergence results for the subdivision operator $\mathcal{S}_m$ in Corollary 4.2.8, for $m = 2, \ldots, 10$. For $m = 2, 4$ and 10, convergence was analysed in Examples 5.2.1 - 5.2.3. Similarly, by applying (4.2.44) in Corollary 4.2.8, together with the results in Table 3.1 and the definition (3.2.3), we may compute the sequence $\{a_{m,j} : j \in \mathbb{Z}\}$ for $m = 3, 5, \ldots, 9$. As in Examples 5.2.1 - 5.2.3, we may then calculate the values $\sum_j |a_{m,3j}|$, $\sum_j |a_{m,3j+1}|$ and $\sum_j |a_{m,3j+2}|$, for $m = 3, 5, \ldots, 9$. These values, together with the results from Examples 5.2.1 - 5.2.3, are compiled in Table 5.1. Observe that the condition

$$\max\left\{\sum_j |a_{m,3j}|, \sum_j |a_{m,3j+1}|, \sum_j |a_{m,3j+2}|\right\} < 1, \tag{5.2.28}$$

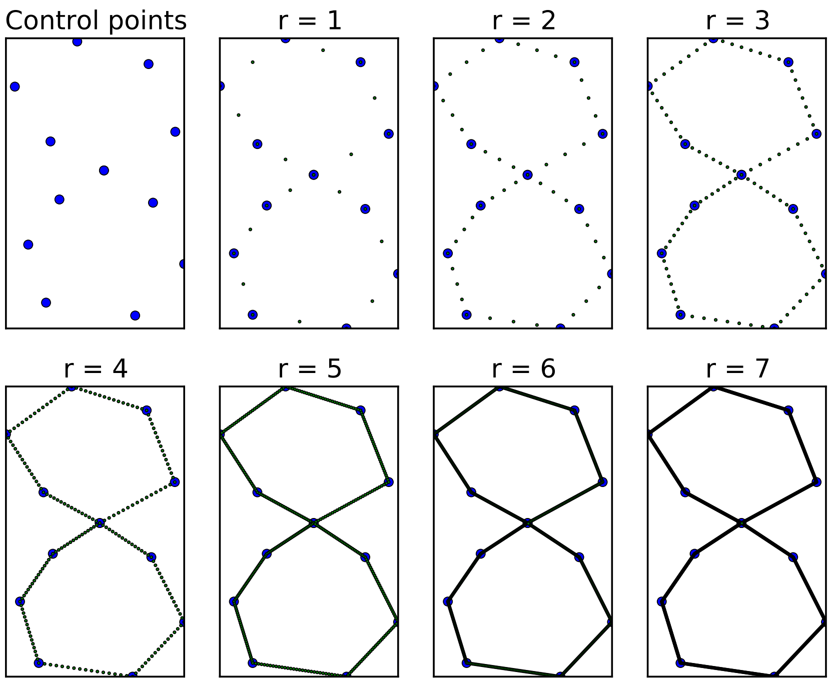
is satisfied in each case, so that we may deduce, from Corollary 4.2.8, that the subdivision operator $\mathcal{S}_m$ provides a convergent interpolatory subdivision scheme, for $m = 2, \ldots, 10$.

m	$\sum_j \lvert a_{m,3j} \rvert$	$\sum_j \lvert a_{m,3j+1} \rvert$	$\sum_j \lvert a_{m,3j+2} \rvert$
2	—	$\frac{1}{3}$	—
3	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{3}$
4	$\frac{1}{9}$	$\frac{25}{81}$	$\frac{1}{9}$
5	$\frac{23}{81}$	$\frac{35}{243}$	$\frac{23}{81}$
6	$\frac{37}{243}$	$\frac{217}{729}$	$\frac{37}{243}$
7	$\frac{565}{2187}$	$\frac{1057}{6561}$	$\frac{565}{2187}$
8	$\frac{389}{2187}$	$\frac{51576}{177147}$	$\frac{389}{2187}$
9	$\frac{128658}{531441}$	$\frac{376}{2187}$	$\frac{128658}{531441}$
10	$\frac{313274}{1594323}$	$\frac{457130}{1594323}$	$\frac{313274}{1594323}$

Table 5.1: Convergence results for the subdivision operator $\mathcal{S}_m$ for $m = 2, \ldots, 10$.

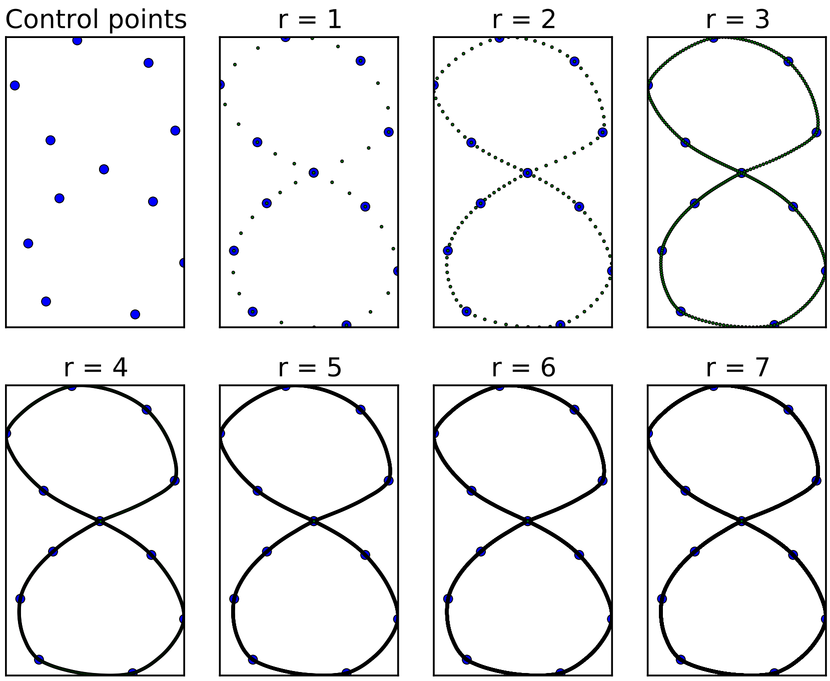


(a) Convergence of ternary interpolatory subdivision scheme, with m=2.

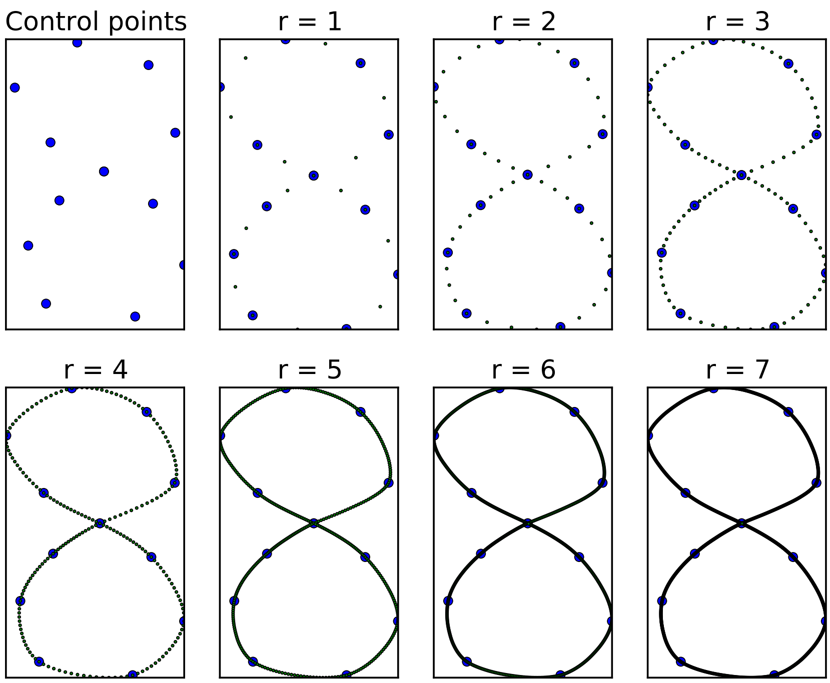


(b) Convergence of binary interpolatory subdivision scheme, with m=2.

Figure 5.1: Convergence of interpolatory subdivision schemes with M=12

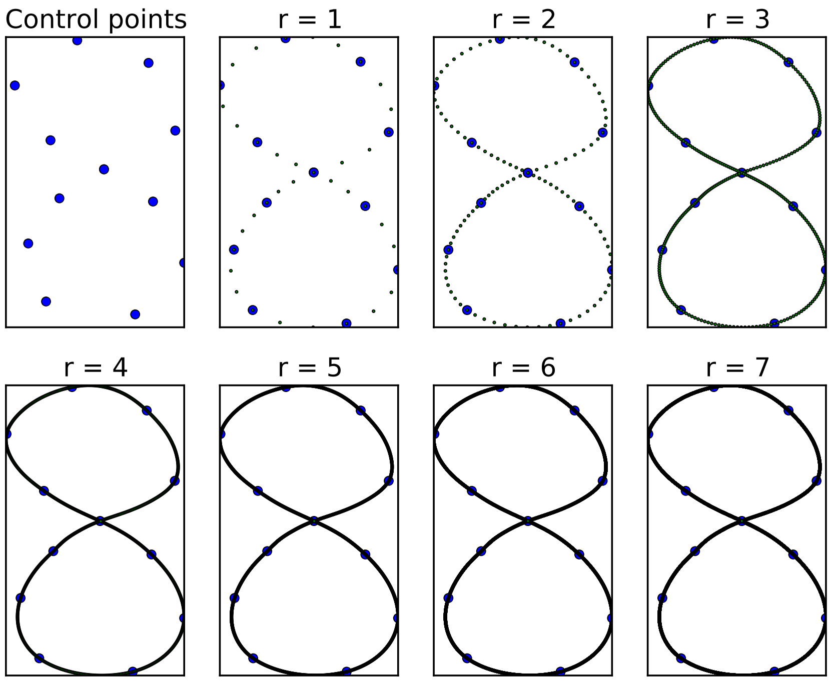


(a) Convergence of ternary interpolatory subdivision scheme, with m=4.

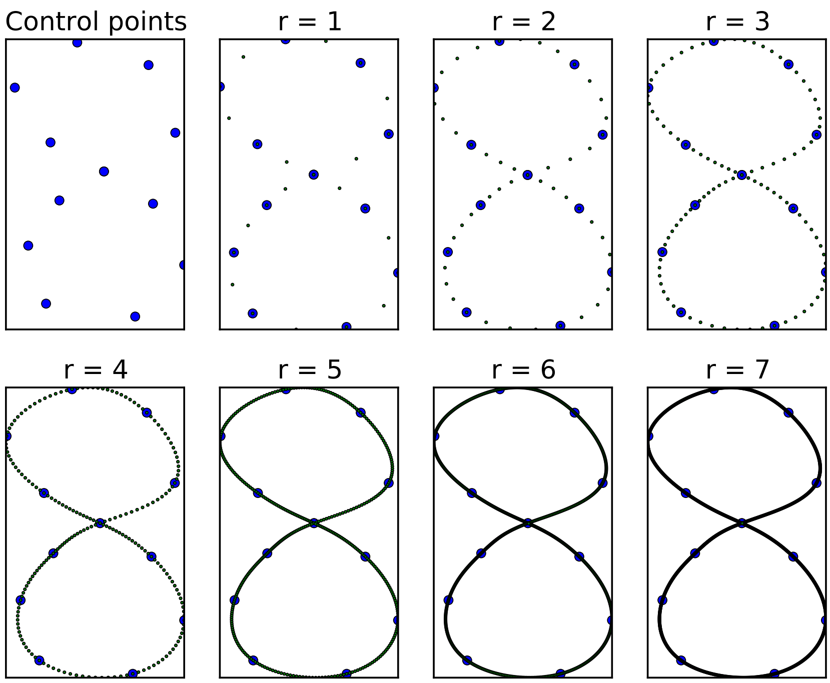


(b) Convergence of binary interpolatory subdivision scheme, with m=4.

Figure 5.2: Convergence of interpolatory subdivision schemes with M=12



(a) Convergence of ternary interpolatory subdivision scheme, with m=10.



(b) Convergence of binary interpolatory subdivision scheme, with m=10.

Figure 5.3: Convergence of interpolatory subdivision schemes with M=12

Chapter 6

Conclusions

In this thesis, we have constructed a symmetric ternary interpolatory subdivision scheme, analogous to the Dubuc-Deslauriers binary subdivision scheme, as introduced in [11] and [10]. We have derived a convergence criterion for this subdivision scheme, which will ensure the existence of an interpolatory refinable limit function, and we have also presented graphical illustrations of the results. As noted in the discussion in this thesis, some of the results were obtained by straightforward adaptations of analogous results in binary subdivision in [1] and [17], while other results are non-trivial extensions of the analogous results in [1] and [17].

The advantage of the ternary scheme derived in this thesis, as opposed to the analogous Dubuc-Deslauriers binary scheme, is that convergence is faster, as can clearly be seen when comparing the graphical results in Figures 5.1 - 5.3. However, the trade-off is that the ternary refinement sequences obtained here are longer than the analogous binary refinement sequences (see Table 8.2.1, p 305 in [1]). As a result, the formulae contained in the ternary scheme are longer and more complicated than in the analogous binary scheme, and thus the algorithm for rendering closed curves takes a longer time to execute.

It is shown in [17] that the analogous Dubuc-Deslauriers binary interpolatory subdivision scheme is convergent for each $m \in \mathbb{N}$. In this thesis, we have derived a convergence criterion on the refinement sequence $\boldsymbol{p}_m$ (in Corollary 4.2.8), and we have shown that the convergence criterion is satisfied for $m = 2, \ldots, 10$ (see Table 5.1). Convergence for all values of $m \in \mathbb{N}$ is to be investigated in further research.

Further investigation is needed on the regularity (or smoothness) of the subdivision limit curve. The Hölder regularity of 2-refinable functions and their corresponding subdivision limit curves are analysed in [1] (see Theorem 6.5.2, p 242, as well as Theorems 8.2.3 and 8.2.4, pp 308-309, for applications in the interpolatory case). As also suggested by the graphs in Figures 5.1 -

5.3, it is our conjecture that the regularity of the 3-refinable functions ϕ_m in Corollary 4.2.8 and their corresponding subdivision limit curves increase with m. We intend to investigate this important issue in further research.

We have presented an analogous ternary scheme of the Dubuc-Deslauriers binary interpolatory subdivision scheme. It would be interesting to see whether the method of construction presented here could also be applied to construct analogous interpolatory subdivision schemes with general refinement factor $k > 3$.

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